**Task 1** (30 points)
Review relevant properties of three-dimensional rotation matrices in terms of transpose, general structure, and, possibly the underlying Lie algebra (what is this?). Verify the properties on examples of your choice (say, rotation matrices which describe rotations about the $z$ axis).

**Task 2** (30 points)
Write (for yourself) a "cheat sheet" that summarizes essential formulas pertaining to ordinary and spherical Bessel functions (denoted as $J_n(x)$ and $j_n(x)$, respectively). The cheat sheet should contain

- defining differential equations,
- integral representations,
- asymptotics for small argument,
- asymptotics for large argument.

Then, write a computer program that plots $J_n(x)$ and $j_n(x)$ for $n = 0, 1, 2, 3$, over relevant range of the argument $x$ (say, $0 < x < 20$). Convince yourself that only Bessel functions of order $n = 0$ are nonvanishing at the origin ($x = 0$).

**Task 3** (20 points)
Define the Hankel function as

$$h_\ell^{(1)}(\rho) = j_\ell(\rho) + \mathrm{i}\, y_\ell(\rho)\,, \tag{1}$$

and show that

$$\mathrm{i}\, k\, j_\ell(k\, r_<)\; h_\ell^{(1)}(k\, r_>) \quad \longrightarrow \frac{1}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}} \tag{2}$$

in the limit $k \to 0$, both by looking up the asymptotics as well as by considering the explicit form of the first few functions, say, for $\ell = 0, 1, 2, 3$.

**Task 4** (20 points)
Tabulate the first few Hankel functions and convince yourself, by looking up the constituent $j_\ell$ and $y_\ell$, that the Hankel functions can be written in terms of $\exp(\mathrm{i}\rho)$ and powers of $\rho$. Show that

$$h_0^{(1)}(\rho) = -\mathrm{i}\,\frac{\mathrm{e}^{\mathrm{i}\rho}}{\rho}\,, \qquad h_1^{(1)}(\rho) = -\frac{\mathrm{e}^{\mathrm{i}\rho}}{\rho} - \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\rho}}{\rho^2}\,, \qquad h_2^{(1)}(\rho) = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\rho}}{\rho} - \frac{3\mathrm{e}^{\mathrm{i}\rho}}{\rho^2} - \frac{3\mathrm{i}\mathrm{e}^{\mathrm{i}\rho}}{\rho^3}\,. \tag{3}$$

Show, by an explicit calculation for the three example cases above, that as $\rho \to +\infty$,

$$h_\ell^{(1)}(\rho) \to -\mathrm{i}\,\frac{\mathrm{e}^{\mathrm{i}(\rho-\ell\pi/2)}}{\rho} = (-\mathrm{i})^{\ell+1}\,\frac{\mathrm{e}^{\mathrm{i}\rho}}{\rho}\,, \qquad \mathrm{e}^{\mathrm{i}(-\ell\pi/2)} = \left(\mathrm{e}^{-\mathrm{i}\pi/2}\right)^\ell = (-\mathrm{i})^\ell\,, \qquad \rho \to \infty\,, \tag{4}$$

i.e., that the Hankel functions of different order only differ by a complex phase in the limit $\rho \to +\infty$.

The tasks are due Thursday, 05–NOV–2026.