Task 1 (50 points)

Look up the formulas for the spherical Neumann functions $y_0(x)$, $y_1(x)$ and $y_2(x)$. By explicit Taylor expansion, show that, for small argument, for the cases $\ell = 0, 1, 2$, one has

$$y_\ell(x) \stackrel{x\to 0}{=} -\frac{(2\ell-1)!!}{x^{\ell+1}}\,. \tag{1}$$

Yes, $(2\ell-1)!!$ in the numerator, not $(2\ell+1)!!$; Eq. (3.26b) in the scriptum has a typographical error. How do you calculate

$$(2\ell-1)!! \quad \text{for} \quad \ell = 0 \quad ? \tag{2}$$

You can use the recurrence relation

$$(2\ell+1)!! = (2\ell+1)\,[(2\ell-1)!] \tag{3}$$

and then set $[(2\ell-1)!] = [(2\ell+1)!!]/(2\ell+1)$. In this way, one calculates the result for $(-1)!!$. Show your work!

Task 2 (50 points)

Verify all intermediate steps in Eq. (3.58) of the lecture notes,

$$\left(\vec{\nabla}^2 + k^2\right) \frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi\epsilon_0 r} = -\frac{1}{\epsilon_0}\,\delta^{(3)}(\vec{r})\,. \tag{4}$$

Show your work! Provide explanations for all intermediate steps. Carefully distinguish between scalar and vector quantities!

Task 3 (50 points)

Simplify the general expression

$$J = \lim_{\epsilon\to 0^+} \int_{r'-\epsilon}^{r'+\epsilon} \mathrm{d}r\, \left(\frac{\partial}{\partial r} f(r) \frac{\partial}{\partial r} g(r_<, r_>)\right) \tag{5}$$

for general functions $f = f(r)$ and $g = g(r_<, r_>)$, as much as possible. Here, $r_< = \min(r_<, r_>)$ and $r_> = \max(r_<, r_>)$.

Recall all intermediate steps that lead to the derivation of Eq. (3.87) of the lecture notes,

$$G_0(\vec{r}-\vec{r}', k) = \frac{\exp\left(\mathrm{i}\,k\,|\vec{r}-\vec{r}'|\right)}{4\pi\epsilon_0\,|\vec{r}-\vec{r}'|} = \frac{1}{\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathrm{i}\,k\, j_\ell\,(k\,r_<)\; h_\ell^{(1)}\,(k\,r_>)\; Y_{\ell m}\,(\theta,\varphi)\; Y_{\ell m}^*\,(\theta',\varphi')\,. \tag{6}$$

Use the result given in Eq. (5), at the decisive point in the derivation.

The tasks are due Thursday, 29–OCT–2026, with a possible extension to Tuesday, 03–NOV–2026.