

Task 1 (unmarked)

Review all notes regarding Cauchy's residue theorem. Then, review all notes regarding the retarded, advanced, and Feynman Green functions of the wave equation.

Task 2 (35 points)

Consider the i th spin matrix Σ^i with matrix elements $(\Sigma^i)^{mn}$,

$$(\Sigma^i)^{mn} = -i\epsilon^{imn}, \quad (1)$$

where the superscripts $i, m, n \in \{1, 2, 3\}$ denote Cartesian indices. Write down the matrices $\Sigma^{i=1}$, $\Sigma^{i=2}$ and $\Sigma^{i=3}$ explicitly. Verify the commutator identity

$$[\Sigma^i, \Sigma^j] = \Sigma^i \cdot \Sigma^j - \Sigma^j \cdot \Sigma^i = i\epsilon^{ijk} \Sigma^k, \quad (2)$$

by considering all combinations that lead to a nonvanishing result ($i, j = 1, 2$, $i, j = 1, 3$, and $i, j = 2, 3$). (The Einstein summation convention is being used.) **Verify that the circular polarization vectors $\hat{e}_{+1}$ and $\hat{e}_{-1}$ are eigenvectors of Σ^3 , and determine the eigenvalues.**

Hint: You may also verify the operator identity (2) more elegantly with reference to certain identities relating the Levi-Civita ϵ tensor to the Kronecker δ .

Task 3 (25 points)

Consider the “dimensionless angular-momentum” operators

$$L^i = -i\epsilon^{ijk} x^j \frac{\partial}{\partial x^k} \quad (3)$$

where the superscripts $i, j, k \in \{1, 2, 3\}$ denote Cartesian indices, and the Einstein summation convention is being used. Verify the commutator identity

$$[L^i, L^j] = L^i L^j - L^j L^i = i\epsilon^{ijk} L^k \quad (4)$$

by considering all combinations that lead to a nonvanishing result ($i, j = 1, 2$, $i, j = 1, 3$, and $i, j = 2, 3$), or, using general analytic formulas with reference to certain identities relating the Levi-Civita ϵ tensor to the Kronecker δ .

Task 4 (25 points)

Calculate the matrix exponential of a rotation about the z axis (at least to second order in θ)

$$\mathbb{R} = \exp(i\theta \Sigma^3) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (5)$$

and show, using a drawing, why the above matrix describes a *passive* rotation by an angle θ about the z axis. *For a successful solution, it is sufficient to verify that the left-hand side, and the right-hand side, of Eq. (5) are equal to second order in their Taylor expansions for small θ .* Then, show, at least to first order in θ , the identity

$$\exp(i\theta L^3) f(\vec{r}) = f(\exp(-i\theta \Sigma^3) \cdot \vec{r}). \quad (6)$$

Task 5 (15 points)

Define the general rotation $\mathbb{R} = \mathbb{R}(\vec{\theta})$ and the two rotations $\mathbb{R}_1 = \mathbb{R}(\vec{\theta}_1)$ and $\mathbb{R}_2 = \mathbb{R}(\vec{\theta}_2)$. Furthermore, define the action of a rotation $\mathbb{R}$ to a function as $(U(\mathbb{R})f)(\vec{r}) \equiv f(\mathbb{R}^{-1}\vec{r})$. Show *in very great detail* (there are only few intermediate steps, but they are difficult and should be spelled out correctly) that

$$U(\mathbb{R}_1) U(\mathbb{R}_2) = U(\mathbb{R}_1 \cdot \mathbb{R}_2). \quad (7)$$