Task 1 (40 points)
You are given the space- and time-dependent vector potentials,

$$\Phi(\vec{r},t) = 0\,, \qquad \vec{A}(\vec{r},t) = \hat{\mathrm{e}}_z \, \frac{E_0}{\omega} \, \cos(\omega\, t - k_x\, x)\,. \tag{1}$$

(a) Calculate the electric and magnetic fields.
(b) Verify that the Maxwell equations are fulfilled, and calculate $\rho(\vec{r},t)$ and $\vec{J}(\vec{r},t)$.
(c) Explain why the given fields describe a monochromatic (what is this?) plane electromagnetic wave.

Task 2 (30 points)
You are given the identity

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{J}\,(\vec{r},t)\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{J}\,(\vec{r},t)\right) - \vec{\nabla}^2 \vec{J}\,(\vec{r},t)\;. \tag{2}$$

Derive this identity on the basis of the Levi-Cività tensor ϵ_{ijk} and the identity $\epsilon_{ijk}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$. The Einstein summation convention is being used.

Task 3 (30 points)
Consider the scalar and vector potentials

$$\Phi(\vec{r},t) = -\vec{\mathcal{E}} \cdot \vec{r}\,, \qquad \vec{A}(\vec{r},t) = \vec{0}\,, \tag{3}$$

where $\vec{\mathcal{E}}$ is a constant electric field. Now, consider the gauge function

$$\Lambda(\vec{r},t) = -t\,(\vec{\mathcal{E}} \cdot \vec{r})\,. \tag{4}$$

Show that the transformed potential are

$$\Phi'(\vec{r},t) = 0\,, \qquad \vec{A}'(\vec{r},t) = -\vec{\mathcal{E}}\,t\,. \tag{5}$$

Colloquially speaking, you have transformed the scalar potential away and converted it into a vector potential! Using Φ and $\vec{A}$, and then Φ' and $\vec{A}'$, calculate the electric and magnetic fields and show that they are equal, *i.e.*, not affected by the gauge transformation. *Please note:* The tasks are algebraically very simple; please show every intermediate step with great care (show work!).

Task 4 (30 points)
(a) Show that a suitable vector potential for a static magnetic field $\vec{B}_0$ is $\vec{A}(\vec{r}) = \frac{1}{2}\,(\vec{B}_0 \times \vec{r})$, *i.e.*, that $\vec{\nabla} \times \vec{A}(\vec{r}) = \vec{B}_0$. Here, $\vec{B}_0$ is a constant and static vector.

(b) Show that, for $\vec{B}_0 = B_z \hat{\mathrm{e}}_z$, the result specializes to

$$\vec{A}(\vec{r}) = -\frac{1}{2}\,B_z\,y\,\hat{\mathrm{e}}_x + \frac{1}{2}\,B_z\,x\,\hat{\mathrm{e}}_y\,. \tag{6}$$

Show that

$$\vec{A}'(\vec{r}) = B_z\,x\,\hat{\mathrm{e}}_y \tag{7}$$

leads to the same $\vec{B}$ field as $\vec{A}$. Then, find the gauge transform function Λ that leads from $\vec{A}$ to $\vec{A}'$.

The tasks are due Tuesday, 15–SEP–2026.