

Task 1 (Unmarked) (Variational Calculus) (i) Write a short essay on the definition of the functional derivative of a function. Consider the action functional

$$S[\Phi] = \int_V d^3r \left[\frac{\epsilon_0}{2} (\vec{\nabla}\Phi(\vec{r}))^2 - \rho(\vec{r})\Phi(\vec{r}) \right], \quad (1)$$

of an electrostatic potential $\Phi = \Phi(\vec{r})$. Show that

$$\frac{\delta S[\Phi]}{\delta\Phi(\vec{r})} = -\epsilon_0 \vec{\nabla}^2 \Phi(\vec{r}) - \rho(\vec{r}). \quad (2)$$

Use either the definition of the functional derivative on the basis of an addition of a Dirac- δ function, or via the addition of the variation $\delta\Phi(\vec{r})$ to the argument function. If you use $\delta\Phi(\vec{r})$, then pay close attention to the boundary conditions imposed on the variation $\delta\Phi(\vec{r})$ when you do partial integrations.

Show and comment every step in your derivation. Interpret the result in terms of the connection of the variational principle, the minimization of the electrostatic field energy, and the Laplace equation.

(ii) In which sense is the operator-valued second functional derivative [to be shown]

$$\frac{\delta^2 S[\Phi]}{\delta\Phi(\vec{r}')\delta\Phi(\vec{r})} = -\epsilon_0 \delta^{(3)}(\vec{r} - \vec{r}') \vec{\nabla}^2 \quad (3)$$

positive, i.e., in which sense can we say that

$$\text{“} \frac{\delta^2 S[\Phi]}{\delta\Phi(\vec{r}')\delta\Phi(\vec{r})} > 0 \text{”?} \quad (4)$$

Hint: Think about the “positivity of a matrix” and a connection of this concept to the positivity of all of its eigenvalues.

(iii) Variations $\delta\Phi(\vec{r})$ are allowed which leave boundary conditions intact. If you were allowed to vary boundary conditions, how could you lower the action indefinitely even in the case of a vanishing charge density? Illustrate your findings by way of a suitable chosen simple physical example.

Task 2 (Unmarked) (Variations in Classical Mechanics) Consider the action S for a classical particle bound in a potential,

$$S[\vec{r}(t)] = \int L(\vec{r}(t), \dot{\vec{r}}(t)) dt. \quad (5)$$

Show that, if L is the usual Lagrange function (kinetic minus potential energy), you get Newton’s equations of motion back, by varying the action.

Now, try to reconcile the definition of the functional derivative in terms of an infinitesimal parameter ϵ and a Dirac- δ function, with an input path $\vec{r} = \vec{r}(t)$ that is vector-valued. Would you have to introduce an arbitrary direction \hat{r}_0 for the variation of the path near $t' = t$, and how would this affect the derivation of the Euler–Lagrange equations?

Task 3 (Unmarked) Reconsider the variational problem of the calculation of the potential in a *cylindrical* capacitor, with a more complex trial potential (with the same boundary conditions as in the lecture)

$$w(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \rho) = w(\rho) = \mathcal{A} + \mathcal{B}\rho + \mathcal{C}\rho^2 + \mathcal{D}\rho^3, \quad w(\rho = b) = 0, \quad w(\rho = c) = V_0. \quad (6)$$

Use the parameters $b = 0.2$ cm and $c = 0.8$ cm and plot the variational solution you obtain. Convince yourself that you obtain an even better approximation to the analytic solution than in the lecture, where we had used a three-parameter variational *ansatz*.
