**Task 1. Dipole Moments. (70 points).** You are given the Cartesian expression for the dipole moment $\vec{p}$ of a charge distribution $\rho = \rho(\vec{r})$, which is $\vec{p} = \int \mathrm{d}^3 r \, \vec{r} \, \rho(\vec{r})$. Furthermore, you are given the general expression for multipole moments $q_{\ell m}$ in spherical coordinates, $q_{\ell m} = \int \rho(\vec{r}) \; r^{\ell} \; Y^*_{\ell m}(\theta, \varphi) \; \mathrm{d}^3 r$.

**(a.)** Based on the **three** conversion formulas (which you can take for granted),

$$q_{1\,1} = -\sqrt{\frac{3}{8\pi}} \; (p_x - \mathrm{i}\, p_y) \,, \qquad q_{1\,-1} = \sqrt{\frac{3}{8\pi}} \; (p_x + \mathrm{i}\, p_y) \,, \qquad q_{10} = \sqrt{\frac{3}{4\pi}} \, p_z \,. \tag{1}$$

determine the entries of the conversion matrix $\mathbb{C} = (C_{ij})_{i,j}$ with $i = 1, 2, 3$ and $j = 1, 2, 3$, *i.e.*, determine the entries $C_{ij}$ in the following matrix

$$\begin{pmatrix} q_{1\,1} \\ q_{1\,-1} \\ q_{10} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \,. \tag{2}$$

**(b.)** Determine the block structure of $\mathbb{C}$. Based on the block structure, calculate the inverse $\mathbb{C}^{-1}$ of the conversion matrix. Determine which four elements of $\mathbb{C}^{-1}$ are zero.

**Task 2. Quadrupole Moments. (70 points).** You are given the Cartesian components of the quadrupole tensor ($\vec{r} = \sum_{i=1}^{3} x_i \, \hat{\mathrm{e}}_i$),

$$Q_{ij} = \int \mathrm{d}^3 r \, \left(3 x_i \, x_j - \delta_{ij} \, \vec{r}^{\,2}\right) \, \rho(\vec{r}) \,, \tag{3}$$

with $i = 1, 2, 3$ and $j = 1, 2, 3$. The element $q_{(\ell=2)\,(m=-2)}$ of the in spherical coordinates is

$$q_{22} = \int \rho(\vec{r}) \; r^2 \; Y^*_{2\,-2}(\theta, \varphi) \; \mathrm{d}^3 r \,, \quad \text{where} \quad Y_{22}(\theta, \varphi) = \left(\frac{15}{32\pi}\right)^{1/2} \; \sin^2\theta \; \mathrm{e}^{-2\mathrm{i}\varphi} \,. \tag{4}$$

Your task is to express the spherical tensor element $q_{22}$ in terms of the elements of the Cartesian $Q_{ij}$ tensor. Show your work! Show every intermediate step with diligence! **Hint:** You might encounter a decomposition of the form $q_{22} = \mathcal{A} \, Q_{11} + \mathcal{B} \, Q_{22} + \mathcal{C} \, Q_{12}$, with the coefficients $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ to be determined. At some point in your calculations, you might use the (somewhat trivial) identity

$$x^2 - y^2 = \frac{2x^2 - y^2 - z^2}{3} - \frac{2y^2 - x^2 - z^2}{3} \,. \tag{5}$$

**Task 3. Dipole–Dipole Interactions. (60 points).** The dipole term in Eq. (5.122) of the lecture notes describes the interaction energy $W$ of a local charge distribution with dipole moment $\vec{p}_1$, located at the origin, with an external electric field $\vec{E}_{\rm ext}(\vec{0})$,

$$W_{\rm dipole} = -\vec{E}_{\rm ext}(\vec{0}) \cdot \vec{p}_1 \,. \tag{6}$$

Now, assume that $\vec{E}_{\rm ext}(\vec{0})$ is the external field, observed at the observation point $\vec{r} = \vec{0}$, generated by a dipole $\vec{p}_2$ located at point $\vec{R}$. **Hence, derive the dipole-dipole interaction energy between dipoles $\vec{p}_1$ and $\vec{p}_2$ a distance $\vec{R}$ apart.** Include a drawing! Show your work! Show every intermediate step with diligence!

**Hint:** First, write the *electrostatic potential* (not field!) generated by $\vec{p}_2$ at the observation point $\vec{r}$. Finally, you will set $\vec{r} = \vec{0}$, but not just yet. Your expression might contain terms $\vec{r} - \vec{R}$. Then, differentiate the electrostatic potential with respect to $\vec{r}$ (not $\vec{R}$!), to get the electric field from the potential at the point $\vec{r}$. As a last step, set $\vec{r} = \vec{0}$. You result might be *not equal to, but proportional to*

$$W_{\rm dipole} \propto \frac{\vec{p}_1 \cdot \vec{p}_2 - 3(\hat{R} \cdot \vec{p}_1)\,(\hat{R} \cdot \vec{p}_2)}{R^3} \,, \qquad \hat{R} = \frac{\vec{R}}{|\vec{R}|} = \frac{\vec{R}}{R} \,. \tag{7}$$

The tasks are due Thursday, 07-MAY-2026. Have fun doing the problems!