The Laplace and Poisson equations in spherical coordinates are encountered in a wide range of problems (electromagnetism, thermodynamics, mechanics of continuous media, and quantum mechanics). As might be expected, the solutions of these equations have been well studied. The solutions to the angular part of the equation are the spherical harmonics. In the lecture, we had denoted these as $Y_{\ell m}(\theta,\varphi)$.

The spherical harmonics are the appropriately normalized products of the associated Legendre polynomials in $\cos\theta$ and the $\exp(\mathrm{i}m\varphi)$,

$$Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos(\theta))\;\mathrm{e}^{\mathrm{i}m\varphi}\,. \tag{1}$$

Task 1 (30 points)
Show that the $Y_{\ell m}(\theta,\varphi)$ are eigenfunctions of the $\vec{L}^2$ operator,

$$\vec{L}^2\,Y_{\ell m}(\theta,\varphi) = \ell\,(\ell+1)\,Y_{\ell m}(\theta,\varphi)\,, \tag{2}$$

and that they have the property that

$$\vec{L}_z\,Y_{\ell m}(\theta,\varphi) = -\mathrm{i}\frac{\partial}{\partial\varphi}\,Y_{\ell m}(\theta,\varphi) = m\,Y_{\ell m}(\theta,\varphi)\,. \tag{3}$$

Task 2 (30 points)
Verify the following properties by way of example, considering the spherical harmonics with $\ell = 0,1$ and $m = -\ell,\ldots,\ell$ (optionally, not for extra credit, but for extra wisdom, consider the case $\ell = 2$ and $m = -2,1,0,1,2$):

$$Y_{\ell\,-m}(\theta,\varphi) = (-1)^m\;Y_{\ell m}(\theta,\varphi)^*\;, \tag{4a}$$

$$Y_{\ell\,m}(\pi-\theta,2\pi-\varphi) = (-1)^\ell\;Y_{\ell\,-m}(\theta,\varphi)\;, \tag{4b}$$

$$Y_{\ell\,m}(\pi-\theta,\pi+\varphi) = (-1)^\ell\;Y_{\ell\,m}(\theta,\varphi)\;. \tag{4c}$$

Task 3 (30 points)
What is the parity transformation? On the sphere, why is the parity transformation implemented through the transformation (make a drawing!)

$$(\theta,\varphi) \to (\pi-\theta,\pi+\varphi)\,. \tag{5}$$

Task 4 (30 points)
Verify the following orthonormality condition by way of example,

$$\int_0^{2\pi}\int_0^{\pi} Y^*_{\ell'\,m'}(\theta,\varphi)\;Y_{\ell m}(\theta,\varphi)\;\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \delta_{\ell\ell'}\;\delta_{mm'}\,, \tag{6}$$

considering the spherical harmonics with $\ell,\ell' = 0,1$, $m = -\ell,\ldots,\ell$ and $m' = -\ell',\ldots,\ell'$.

The tasks are due Thursday, 23-APR-2026. Have fun doing the problems!