Task 1 (20 points). Show that for a scalar field $\Psi = \Psi(\vec{r})$,

$$\oint_{\partial A} \Psi(\vec{s})\, \mathrm{d}\vec{s} = \int_A \mathrm{d}\vec{A} \times \vec{\nabla}\Psi(\vec{r}) = \int_A \hat{n} \times \vec{\nabla}\Psi(\vec{r})\, \mathrm{d}A\,. \tag{1}$$

The latter two expressions are equal by definition, the first equality is to be shown. One possible method of proof is to apply Stokes's theorem to a vector field $\vec{V}(\vec{s}) = \vec{b}\,\Psi(\vec{s})$, where $\vec{b}$ is an arbitary constant vector. Please use this method.

Task 2 (30 points). Define

$$||\vec{r}|| = \sqrt{x^2 + y^2 + z^2}\,, \qquad x_1 = x\,, \quad x_2 = y\,, \quad x_3 = z\,. \tag{2}$$

Show that for all $i, j = 1, 2, 3$, one has the relation

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \frac{1}{||\vec{r}||} = \Theta(||\vec{r}||) \left\{ \frac{3 x_i x_j}{||\vec{r}||^5} - \frac{\delta_{ij}}{||\vec{r}||^3} \right\} - \frac{4\pi}{3} \delta_{ij}\, \delta^{(3)}(\vec{r})\,. \tag{3}$$

Here, $\Theta(x)$ is the Heaviside step function, defined so that $\Theta(x) = 0$ for $x \leq 0$, and $\Theta(x) = 1$ for $x > 0$. Why is this result compatible with the defining equation for the Green function of the Poisson equation in three dimensions? Write a little essay on what happens when you let $i = j$ and sum over i from 1 to 3.

Task 3 (30 points). Derive the wave equations

$$\left(\vec{\nabla}^{\,2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E} = \vec{0}\,, \qquad \left(\vec{\nabla}^{\,2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{B} = \vec{0}\,, \tag{4}$$

for the electric and magnetic fields in a source-free region, $\rho = 0$ and $\vec{J} = 0$, by working in the SI mksA unit system in all intermediate steps.

Task 4 (60 points). Investigate the Ampere–Maxwell law. **(a.)** Take the divergence of both sides of the Ampere–Maxwell law,

$$\vec{\nabla} \times \vec{B}(\vec{r}, t) = \mu_0\, \vec{J}(\vec{r}, t) + \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}(\vec{r}, t)\,, \tag{5}$$

and show that you obtain the time derivative of Gauss's law. **(b.)** Give a reason why the Ampere law, which would assert that $\vec{\nabla} \times \vec{B}(\vec{r}, t)$ is equal to $\mu_0\, \vec{J}(\vec{r}, t)$ [and ignores the second term on the right-hand side of Eq. (5)], must be physically inconsistent upon closer inspection. **(c.)** Show, using the integral form of the Ampere–Maxwell law, that the term proportional to $\partial_t \vec{E}$ restores the consistency of the equation, when considering a deformed contour in between parallel capacitor plates which are being charged through a current I. Please note: A corresponding hint has been given in the lecture.

The tasks are due Thursday, 09-APR-2026. Have fun doing the problems!