Task 1 (30 points)
In the lecture, we had encountered the following integral representations of the retarded and advanced Green's function of the one-dimensional Poisson equation,

$$g_R(t-t') = -\int \frac{d\omega}{2\pi} \, \frac{e^{-i\omega(t-t')}}{(\omega + i\epsilon)^2} \,, \qquad g_A(t-t') = -\int \frac{d\omega}{2\pi} \, \frac{e^{-i\omega(t-t')}}{(\omega + i\epsilon)^2} \,, \tag{1}$$

where the limit $\epsilon \to 0^+$ is understood *after* all integrals have been evaluated. The presence of the infinitesimal imaginary parts in the denominators means that singularities are displaced from the real axis, infinitesimally, so that, *e.g.*, for the retarded Green's function, the pole is at $\omega = -i\epsilon$.

Instead(!!!) of introducing the infinitesimal imaginary parts, find integration contours C_R and C_A, which avoid the point ω by encircling the pole at $\omega = 0$ either infinitesimally above or below, so that

$$g_R(t-t') = -\int_{C_R} \frac{d\omega}{2\pi} \, \frac{e^{-i\omega(t-t')}}{\omega^2} \,, \qquad g_A(t-t') = -\int_{C_A} \frac{d\omega}{2\pi} \, \frac{e^{-i\omega(t-t')}}{(\omega + i\epsilon)^2} \,. \tag{2}$$

Evaluate the integrals in Eq. (2) on the basis of the singularities (residues) encountered at exact $\omega = 0$.

Task 2 (30 points)
Show by explicit differentiation that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \ln\left(\frac{\sqrt{x^2+y^2}}{a}\right) = 0 \,, \tag{3}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \frac{1}{\sqrt{x^2+y^2+z^2}} = 0 \,, \tag{4}$$

provided $x \neq 0$, $y \neq 0$, $z \neq 0$ and $\xi \neq 0$. How is the derived result related to the Green functions of the Poisson equation in two, three and four dimensions? (Hint: For the first and the second of the above three equations, you may use your lecture notes and an appropriate notation of your choice, e.g., $r \equiv ||\vec{r}|| = \sqrt{x^2+y^2+z^2}$.)

Task 3 (30 points)
With the use of Gauss's theorem (divergence theorem), determine the prefactors which lead to solutions of the Poisson equations in two, three and four dimensions,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) g(x,y) = \delta^{(2)}(x,y) \,, \tag{5}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) g(x,y,z) = \delta^{(3)}(x,y,z) \,. \tag{6}$$

(Hint: You should formulate the divergence theorem in such a way that it is amenable to a generalization to four dimension. How would you paramerize a unit sphere imbedded in three dimensions? How would you paramerize a unit sphere imbedded in four dimensions?)

Task 4 (30 points)
Calculate the Green function of the Poisson equation in three dimensions,

$$g(\vec{r} - \vec{r}') = -\frac{1}{4\pi|\vec{r} - \vec{r}'|} \tag{7}$$

by Fourier transforming to wave vector space, and backtransforming to position space. Any other solution will result in zero points.

The tasks are due Thursday, 12–MAR–2025. Have fun doing the problems!