Task 1 (50 points)
Verify the correctness of the first two Taylor series terms in the expansion of the Γ function for small argument,

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\,, \qquad \gamma_E = 0.577\,215\,664\,\ldots\,, \tag{1}$$

where the γ_E is the Euler constant and the Γ function is defined as $\Gamma(x) = \int_0^\infty t^{x-1}\,\mathrm{e}^{-t}\,\mathrm{d}t$. Use **(a)** an explicit analytic calculation of the terms. To this end, first show, by partial integration, that

$$\Gamma(\epsilon) = \frac{1}{\epsilon}\int_0^\infty t^\epsilon\,\mathrm{e}^{-t}\,\mathrm{d}t\,. \tag{2}$$

Then, do another partial integration, and expand in ϵ. Use the formula $t^\epsilon = 1 + \epsilon\,\ln(t) + \mathcal{O}(\epsilon^2)$ (or similar) and consider the integral representation

$$\gamma_E = \int_0^\infty \mathrm{d}t\, t\,[1 - \ln(t)]\,\mathrm{e}^{-t}\,. \tag{3}$$

(b) Use a simple plot of the function, possibly with the term $1/\epsilon$ subtracted, and verify your result graphically. It is enough to convince yourself that, graphically, $\lim_{\epsilon\to 0}\left(\Gamma(\epsilon) - \frac{1}{\epsilon}\right) = -\gamma_E$. A simple plot, for a set of numerically small values of ϵ, will be sufficient.

Task 2 (50 points). Verify the Mittag–Leffler expansion of the cosecans,

$$\frac{1}{\sin(z)} = \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{z - n\pi} = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2\,(-1)^k\,z}{z^2 - (k\pi)^2}\,, \qquad z \neq n\pi\,, \qquad n \in \mathbb{Z}\,, \tag{4}$$

for three complex example cases of your choice, say,

$$z = z_1 = 10 + 5\mathrm{i}\,, \qquad z = z_2 = 40 + 7\mathrm{i}\,, \qquad z = z_3 = 100.3\,, \tag{5}$$

by evaluating the complex sine function on the left-hand side, and comparing to a numerically calculated sum for the right-hand side. You may use a computer symbolic program of your choice, and replace the upper limit (formally, infinity) of the k-summation by some maximum $k_{\max}$, which you should choose large enough so that at least 3 significant digits can be numerically verified.

Task 3 (50 points)
Show that $g(x - x')$ is a Green function of the one-dimensional Poisson equation,

$$g(x - x') = \frac{|x - x'|}{2}\,, \qquad \frac{\partial^2}{\partial x^2} g(x - x') = \delta(x - x')\,, \tag{6}$$

and that

$$\tilde{g}(x - x') = \Theta(x - x')\,(x - x') \tag{7}$$

also is a valid Green function of the one-dimensional Poisson equation,

$$\frac{\partial^2}{\partial x^2}\tilde{g}(x - x') = \delta(x - x')\,. \tag{8}$$

Also, show that

$$f(x - x') = g(x - x') - \tilde{g}(x - x') \tag{9}$$

is a solution of the homogeneous equation.

The tasks are due Thursday, 05–MAR-2026. Have fun doing the problems!