This sheet is a derivation of certain useful identities on Bessel functions.

It contains a number of missing steps which you could fill in, in the sense of an ungraded exercise.

The missing steps really are quite straightforward to fill in.

Step #1: Bessel's differential equation is

$$x^2\, y''(x) + x\, y'(x) + (x^2 - n^2)\, y(x) = 0\,. \tag{1}$$

The independent solutions are $J_n(x)$ and $Y_n(x)$. The solution regular at the origin is $J_n(x)$. Show that for an arbitrary scale parameter λ, the function $J_n(\lambda\, x)$ fulfills

$$x \frac{\mathrm{d}}{\mathrm{d}x}\left[x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\lambda\, x)\right] + \left(\lambda^2 x^2 - n^2\right)\, J_n(\lambda\, x) = 0\,. \tag{2}$$

Now assume that λ and μ are two distinct zeros of the Bessel function $J_n(x)$, i.e.,

$$J_n(\lambda) = J_n(\mu) = 0\,, \qquad \lambda \neq \mu\,, \qquad J_n(\lambda\, x) = J_n(\mu\, x) = 0 \qquad \text{for } x = 1 \tag{3}$$

Multiplying Eq. (2) by $J_n(\mu\, x)/x$, show that the following equation holds,

$$J_n(\mu\, x) \frac{\mathrm{d}}{\mathrm{d}x}\left[x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\lambda\, x)\right] + \frac{\lambda^2 x^2 - n^2}{x}\, J_n(\mu\, x)\, J_n(\lambda\, x) = 0\,. \tag{4}$$

Show (how?) that the following relation also holds,

$$J_n(\lambda\, x) \frac{\mathrm{d}}{\mathrm{d}x}\left[x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\mu\, x)\right] + \frac{\mu^2 x^2 - n^2}{x}\, J_n(\mu\, x)\, J_n(\lambda\, x) = 0\,. \tag{5}$$

Furthermore, manipulating Eqs. (4) and (5), show that (how?)

$$J_n(\mu\, x) \frac{\mathrm{d}}{\mathrm{d}x}\left[x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\lambda\, x)\right] - J_n(\lambda\, x) \frac{\mathrm{d}}{\mathrm{d}x}\left[x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\mu\, x)\right] + \left(\lambda^2 - \mu^2\right) x J_n(\mu\, x)\, J_n(\lambda\, x) = 0\,. \tag{6}$$

Now, work with Eq. (6) and show that it can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[J_n(\mu\, x)\, x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\lambda\, x)\right] - \frac{\mathrm{d}}{\mathrm{d}x}\left[J_n(\lambda\, x)\, x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\mu\, x)\right] + \left(\lambda^2 - \mu^2\right) x J_n(\mu\, x)\, J_n(\lambda\, x) = 0\,. \tag{7}$$

Finally, integrate over $x \in (0, 1)$, use the fact that $J_n(\lambda) = J_n(\mu) = 0$, and show that

$$\int_0^1 \mathrm{d}x\, x\, J_n(\mu\, x)\, J_n(\lambda\, x) = 0\,, \qquad \lambda \neq \mu\,, \qquad J_n(\lambda) = J_n(\mu) = 0\,, \tag{8}$$

In doing so, treat the case $n = 0$ separately; it requires special attention at the lower limit of integration because $J_n(0) = \delta_{n0}$ for $n \in \mathbb{N}_0$. However, you can use the known fact that $J_n'(0) = 0$. In general,

$$J_n(x) = \frac{x^n}{2^n\, n!} + \mathcal{O}(x^{n+2})\,, \qquad x \to 0\,. \tag{9}$$

Step #2: From the recursion of the Bessel function

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}\, J_n(x) \tag{10}$$

and the formula

$$\frac{\mathrm{d}}{\mathrm{d}x} J_n(x) = \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right) \tag{11}$$

conclude that (how?), at a zero of the Bessel function,

$$\frac{\mathrm{d}}{\mathrm{d}x} J_n(x) = -J_{n+1}(x) \qquad \text{for } J_n(x) = 0\,. \tag{12}$$

In the formula given in Eq. (7),

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[J_n(\mu\, x)\, x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\lambda\, x)\right] - \frac{\mathrm{d}}{\mathrm{d}x}\left[J_n(\lambda\, x)\, x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\mu\, x)\right] + \left(\lambda^2 - \mu^2\right) x J_n(\mu\, x)\, J_n(\lambda\, x) = 0\,, \tag{13}$$

set $\mu = \lambda + \epsilon$ and integrate over $x \in (0,1)$ to obtain (how?)

$$\left[J_n((\lambda+\epsilon))\, \lambda J_n'(\lambda)\right] + \left(\lambda^2 - (\lambda+\epsilon)^2\right) \int_0^1 \mathrm{d}x\, x\; J_n((\lambda+\epsilon)\, x)\; J_n(\lambda\, x) = 0\,, \tag{14}$$

Expand to first order in ϵ to obtain (how?)

$$\epsilon\, \lambda \left[J_n'(\lambda)\right]^2 - 2\,\lambda\,\epsilon \int_0^1 \mathrm{d}x\, x\; \left[J_n(\lambda\, x)\right]^2 = 0\,, \tag{15}$$

and thus show that

$$\int_0^1 \mathrm{d}x\, x\; \left[J_n(\lambda\, x)\right]^2 = \frac{1}{2}\left[J_{n+1}(\lambda)\right]^2\,. \tag{16}$$

This completes the result (8) for the special case $\lambda = \mu$.

As a last step, perform the scale transformation

$$\lambda \to \tilde{\lambda}\, a, \qquad x \to \rho/a\,, \tag{17}$$

to obtain

$$\int_0^a \mathrm{d}\rho\, \rho\; \left[J_n(\tilde{\lambda}\, \rho)\right]^2 = \frac{1}{2}\, a^2\, \left[J_{n+1}(\tilde{\lambda}\, a)\right]^2\,, \qquad \text{for } J_n(\tilde{\lambda}\, a) = 0\,. \tag{18}$$

Step #3: Start once more from Eq. (7), but with the replacements $\lambda \to a$, and $\mu \to b$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[J_n(b\,x)\,x\frac{\mathrm{d}}{\mathrm{d}x}J_n(a\,x)\right] - \frac{\mathrm{d}}{\mathrm{d}x}\left[J_n(a\,x)\,x\frac{\mathrm{d}}{\mathrm{d}x}J_n(b\,x)\right] + \left(a^2 - b^2\right) x J_n(a\,x)\,J_n(b\,x) = 0\,. \tag{19}$$

Verify, by looking at your favorite literature reference, that

$$J_n(\rho) \sim \sqrt{\frac{2}{\pi\,\rho}}\,\sin\left(\rho - \frac{(n-1/2)\,\pi}{2}\right)\,, \qquad \rho \to \infty\,. \tag{20}$$

This implies that $\rho = \infty$ is a zero of the Bessel function. Integrating, thus, Eq. (19) within the interval $x \in (0, \infty)$, show that (how?)

$$\int_0^\infty \mathrm{d}x\,x\,J_n(a\,x)\,J_n(b\,x) = 0\,, \qquad a \neq b\,. \tag{21}$$

You may have to treat the case $n = 0$ separately and observe that the slope of $J_0(x)$ vanishes at $x = 0$.

Now treat the limit $a \to b$. The only region which can sizeably contribute to the integral in this limit is the one for very large x; otherwise only a very small displacement $a = b + \epsilon$ will lead to a vanishing integral. Write the asymptotics (20) as an exponential,

$$J_n(a\,x) \sim \frac{1}{2\,\mathrm{i}}\sqrt{\frac{2}{\pi\,a\,x}}\left\{\exp\left[\mathrm{i}\left(a\,x - \frac{(n-1/2)\,\pi}{2}\right)\right] - \exp\left[-\mathrm{i}\left(a\,x - \frac{(n-1/2)\,\pi}{2}\right)\right]\right\}\,, \qquad x \to \infty\,, \tag{22}$$

$$J_n(b\,x) \sim \frac{1}{2\,\mathrm{i}}\sqrt{\frac{2}{\pi\,b\,x}}\left\{\exp\left[\mathrm{i}\left(b\,x - \frac{(n-1/2)\,\pi}{2}\right)\right] - \exp\left[-\mathrm{i}\left(b\,x - \frac{(n-1/2)\,\pi}{2}\right)\right]\right\}\,, \qquad x \to \infty\,. \tag{23}$$

Show that (how?) the only relevant terms in the integrand $x\,J_n(a\,x)\,J_n(b\,x)$ in Eq. (21) are given by the following replacement,

$$\begin{aligned} x\,J_n(a\,x)\,J_n(b\,x) &\to x\left(\frac{1}{2\,\mathrm{i}}\right)^2\sqrt{\frac{2}{\pi\,a\,x}}\sqrt{\frac{2}{\pi\,b\,x}}\,\left\{-\exp\left(\mathrm{i}\,(a-b)\;x\right) - \exp\left(-\mathrm{i}\,(a-b)\;x\right)\right\} \\ &\to \frac{1}{2\pi\,\sqrt{a\,b}}\,\left\{\exp\left(\mathrm{i}\,(a-b)\;x\right) + \exp\left(-\mathrm{i}\,(a-b)\;x\right)\right\}\,. \end{aligned} \tag{24}$$

Symmetrize the integrand for $a \to b$ within the interval $x \in (-\infty, \infty)$ to show that (how?)

$$\begin{aligned} \int_0^\infty \mathrm{d}x\,x\,J_n(a\,x)\,J_n(b\,x) &= \frac{1}{2}\int_{-\infty}^\infty \mathrm{d}x\,\frac{1}{2\pi\,\sqrt{a\,b}}\,\left\{\exp\left(\mathrm{i}\,(a-b)\;x\right) + \exp\left(-\mathrm{i}\,(a-b)\;x\right)\right\} \\ &= \frac{1}{2\pi\,\sqrt{a\,b}}\,2\pi\delta(a-b) = \frac{1}{\sqrt{a\,b}}\,\delta(a-b) = \frac{1}{a}\,\delta(a-b)\,, \qquad a \to b\,. \end{aligned} \tag{25}$$

Furthermore, for $a \neq b$, this relation actually reduces to Eq. (21) for $a \neq b$, so it is valid for all a and b.

Step #4: Start from the relation

$$\int_0^\infty \mathrm{d}x\, x\, J_n(a\,x)\, J_n(b\,x) = \frac{1}{\sqrt{a\,b}}\,\delta(a-b)\,. \tag{26}$$

Spherical Bessel functions are defined as

$$j_\ell(\rho) = \sqrt{\frac{\pi}{2\,\rho}}\,J_{\ell+1/2}(\rho)\,. \tag{27}$$

Show that you can rewrite Eq. (26) as follows (how do you have to identify the variables?),

$$\int_0^\infty \mathrm{d}x\, x^2 \left(\sqrt{\frac{\pi}{2\,k\,x}}\,J_{\ell+1/2}(k\,x)\right)\left(\sqrt{\frac{\pi}{2\,k'\,x}}\,J_{\ell+1/2}(k'\,x)\right) = \left(\sqrt{\frac{\pi}{2}}\right)^2 \frac{1}{k\,k'}\,\delta(k-k')\,. \tag{28}$$

Finally, show that (how?)

$$\int_0^\infty \mathrm{d}x\, x^2\, j_\ell(k\,x)\, j_\ell(k'\,x) = \frac{\pi}{2\,k\,k'}\,\delta(k-k')\,. \tag{29}$$

The task is voluntary, unmarked, and for Geeks, but is highly recommended.