Task 1 (40 points) (Trigonometric Integrals) By writing the trigonometric function as exponentials, show that, for m and m' integer, one has the relation

$$\int_0^\pi \mathrm{d}\xi \, \sin(m\xi) \sin(m'\xi) = \frac{\pi}{2} \delta_{mm'} \,. \tag{1}$$

Task 2 (40 points) (Bessel Functions I) Start from the defining equation of the ordinary Bessel function, $J_m(x)$. Also, look up the defining differential equation for the spherical Bessel function, $j_m(x)$. Write $j_m(x)$ in terms of $J_m(x)$, with the prefactor $\sqrt{\pi/(2x)}$, and convince yourself that, if you plug this relation into the defining equation for the spherical Bessel function, you obtain the ordinary Bessel differential equation.

Task 3 (40 points) (Bessel Functions II) Verify the two asymptotic relations

$$J_m(x) \sim \frac{x^m}{(2m)!!} \quad (\text{for } x \to 0) \,, \qquad J_m(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left[x - \frac{\pi}{2} \left(m - \frac{1}{2} \right) \right] \quad (\text{for } x \to \infty) \,, \tag{2}$$

by way of example, using graphical and numerical computer algebraic programs of your choice, choosing for m suitable integer values (perhaps, not too large). One chooses a domain of positive x, i.e., $x > 0$. Two plots will be made, for the regions $x \ll m$ and $x \gg m$.

Task 4 (40 points) (Bessel Functions II) Verify the recursion relations $J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x)$, and $J_{m-1}(x) - J_{m+1}(x) = 2J'_m(x)$, by way of example, using graphical and numerical computer algebraic programs of your choice, choosing for m suitable integer values (perhaps, not too large). One selects the regime $x > 0$, and, perhaps, $x \sim m$.

Task 5 (20 points) (Laplace and Helmholtz Equations) Summarize the defining differential equations of the ordinary and spherical Bessel functions. Write an essay on the two questions: (i) How does the ordinary Bessel function enter the solution of the Laplace equation in cylindrical coordinates? (ii) How does the spherical Bessel function enter the solution of the Helmholtz equation in spherical coordinates?

Task 6 (100 extra points, hard) (Boundary–Value Problem) Define $\rho = \sqrt{x^2 + y^2}$ as the distance from the z axis. For the Laplace equation $\vec{\nabla}^2 \Phi(\vec{r}) = 0$, you are given a boundary-value problem for a cylinder, whose symmetry axis coincides with the z axis, and which extends from $z = -L/2$ to $z = L/2$ (i.e., the cylinder has a length L). The radius of the cylinder is a. The walls of the cylinder are held at potential $\Phi = 0$ (grounded), and both end caps held at the electrostatic potential

$$\Phi(x, y, z = \pm \tfrac{1}{2} L) = \Phi_0 \left(1 - \frac{x^2 + y^2}{a^2} \right) = \Phi_0 \left(1 - \frac{\rho^2}{a^2} \right) \tag{3}$$

Explain why a suitable *ansatz* for the solution of this boundary-value problem is

$$\Phi(x, y, z) = \sum_{mn} c_{mn} \, J_m \left(\xi_{mn} \frac{\rho}{a} \right) \cosh \left(\xi_{mn} \frac{z}{a} \right) \exp(\mathrm{i} m \varphi) \,. \tag{4}$$

Here, ξ_{mn} is the nth zero of the mth Bessel function,

$$J_m(\xi_{mn}) = 0 \,, \qquad m = 0, 1, 2, \dots \,, \qquad n = 1, 2, 3, \dots \,, \tag{5}$$

Convince yourself that the only contributing terms have $m = 0$, and, on the basis of the theory of Bessel functions, determine c_{mn} and simplify the result as much as possible. (Only terms with $m = 0$ will contribute.) If you feel ambitious, plot the resultant series (the first terms in the sum over n) for the parameters $L = a = 1$, and exhibit the convergence toward the boundary condition.

The tasks are due on Thursday, 24–APR–2025.