Task 1 (20 points) (Recap and Reading Ahead) The semester is coming to a close. Please study, understand, digest and incorporate all the material in the lecture covered so far. In particular, try to verify all the steps leading to the residue terms in the calculation of the potential outlined in Sec. 6.6 of the lecture notes. *An written affidavit confirming that the task has been completed is sufficient for the assignment of the points.*

Task 2 (70 points) (Variational Calculus) *(i)* Write a short essay on the definition of the functional derivative of a function. Consider the action functional

$$S[\Phi] = \int_V \mathrm{d}^3 r \left[\frac{\epsilon_0}{2} \left(\vec{\nabla} \Phi(\vec{r}) \right)^2 - \rho(\vec{r})\, \Phi(\vec{r}) \right] , \tag{1}$$

of an electrostatic potential $\Phi = \Phi(\vec{r})$. Show that

$$\frac{\delta S[\Phi]}{\delta \Phi(\vec{r})} = -\epsilon_0 \vec{\nabla}^2 \Phi(\vec{r}) - \rho(\vec{r}) \,. \tag{2}$$

Use either the definition of the functional derivative on the basis of an addition of a Dirac-δ function, or via the addition of the variation $\delta\Phi(\vec{r})$ to the argument function. If you use $\delta\Phi(\vec{r})$, then pay close attention to the boundary conditions imposed on the variation $\delta\Phi(\vec{r})$ when you do partial integrations. *Show and comment every step in your derivation.* Interpret the result in terms of the connection of the variational principle, the minimization of the electrostatic field energy, and the Laplace equation.

(ii) In which sense is the operator-valued second functional derivative [to be shown]

$$\frac{\delta^2 S[\Phi]}{\delta \Phi(\vec{r}\,')\, \delta \Phi(\vec{r})} = -\epsilon_0\, \delta^{(3)}(\vec{r} - \vec{r}\,')\, \vec{\nabla}^2 \tag{3}$$

positive, i.e., in which sense can we say that

$$\text{“}\frac{\delta^2 S[\Phi]}{\delta \Phi(\vec{r}\,')\, \delta \Phi(\vec{r})} > 0\text{”}\ ? \tag{4}$$

Hint: Think about the “positivity of a matrix” and a connection of this concept to the positivity of all of its eigenvalues.

(iii) Variations $\delta\Phi(\vec{r})$ are allowed which leave boundary conditions intact. If you were allowed to vary boundary conditions, how could you lower the action indefinitely even in the case of a vanishing charge density? Illustrate your findings by way of a suitable chosen simple physical example.

Task 3 (40 points) Reconsider the variational problem of the calculation of the potential in a *cylindrical* capacitor, with a more complex trial potential (with the same boundary conditions as in the lecture)

$$w(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \rho) = w(\rho) = \mathcal{A} + \mathcal{B}\,\rho + \mathcal{C}\,\rho^2 + \mathcal{D}\,\rho^3 \,, \qquad w(\rho = b) = 0 \,, \qquad w(\rho = c) = V_0 \,. \tag{5}$$

Use the parameters $b = 0.2\,\mathrm{cm}$ and $c = 0.8\,\mathrm{cm}$ and plot the variational solution you obtain. Convince yourself that you obtain an even better approximation to the analytic solution than in the lecture, where we had used a three-parameter variational *ansatz*.

The tasks are due Thursday, 08–MAY–2025.