Task 1 (30 points)
Define the angular momentum operator as

$$\vec{L} = -\mathrm{i}\,(\vec{r} \times \vec{\nabla})\,. \tag{1}$$

Starting from Cartesian coordinates, where

$$L_z = -\mathrm{i}\left(x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial x}\right)\,, \tag{2}$$

use the chain rule to show that

$$L_z = -\mathrm{i}\,\frac{\partial}{\partial \varphi}\,, \tag{3}$$

in spherical coordinates.

Task 2 (20 points)
Define a suitable "average over the solid angle" for an expression $X = X(\theta, \varphi)$,

$$\langle X(\theta,\varphi)\rangle = \frac{1}{\int \mathrm{d}\Omega} \int \mathrm{d}\Omega\, X(\theta,\varphi) = \frac{1}{4\pi} \int \mathrm{d}\Omega\, X(\theta,\varphi)\,, \tag{4}$$

and show that

$$\langle \cos^2(\theta)\rangle = \tfrac{1}{3}\,, \tag{5a}$$
$$\langle \sin^2(\theta)\rangle = \tfrac{2}{3}\,, \tag{5b}$$
$$\langle \sin^2(\theta)\,\cos^2(\varphi)\rangle = \tfrac{1}{3}\,, \tag{5c}$$
$$\langle \sin^2(\theta)\,\sin^2(\varphi)\rangle = \tfrac{1}{3}\,. \tag{5d}$$

Interpret your results in terms of the relation

$$\left\langle \frac{x^2}{r^2} \right\rangle = \left\langle \frac{y^2}{r^2} \right\rangle = \left\langle \frac{z^2}{r^2} \right\rangle = \frac{1}{3}\,. \tag{6}$$

This relation illustrates the "equipartition" between the x, y, and z coordinates on the unit sphere.

The tasks are due Tuesday, 08–APR–2025.