Task 1 (60 points)
Consider the Poisson equation (in three dimensions) for a charge distribution composed of three point charges,

$$\vec{\nabla}^2 \, \Phi(\vec{r}) = -\frac{1}{\epsilon_0} \, \rho(\vec{r}) \,, \qquad \rho(\vec{r}) = q_1 \, \delta^{(3)}(\vec{r} - \vec{r}_1) + q_2 \, \delta^{(3)}(\vec{r} - \vec{r}_2) + q_3 \, \delta^{(3)}(\vec{r} - \vec{r}_3) \,. \tag{1}$$

Symbols are defined as in the lecture. The three point charges are places at $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$.

(a.) Using the Green function of the three-dimensional Poisson equation, calculate the **electrostatic potential** $\Phi(\vec{r})$ and the electric field $\vec{E}(\vec{r})$,

$$\Phi(\vec{r}) = -\frac{1}{\epsilon_0} \int \mathrm{d}^3 r' \, g(\vec{r} - \vec{r}\,') \, \rho(\vec{r}\,') \,, \qquad \vec{E}(\vec{r}) = -\vec{\nabla} \Phi(\vec{r}) \,, \tag{2}$$

by finding an analytic expression involving the parameters q_1, q_2, q_3, $\vec{r}$, $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$. **When calculating the expression** $\int \mathrm{d}^3 r' \, g(\vec{r} - \vec{r}\,') \, \rho(\vec{r}\,')$**, carefully distiguish independent arguments of functions, integration variables, and parameters!!!**

(b.) Assume $q_1 = q_2 = q_3 = 0.04\,\mathrm{C}$, $\vec{r}_1 = \vec{0}$, $\vec{r}_2 = (3.5\,\mathrm{m})\,\hat{\mathrm{e}}_x$, and $\vec{r}_3 = (1.5\,\mathrm{m})\,\hat{\mathrm{e}}_y$. Calculate the quantities

$$\Phi(\vec{r}_a) = ?\,, \qquad \vec{E}(\vec{r}_a) = ?\,, \qquad \vec{r}_a = (1.3\,\mathrm{m})\,\hat{\mathrm{e}}_x + (1.5\,\mathrm{m})\,\hat{\mathrm{e}}_y + (0.1\,\mathrm{m})\,\hat{\mathrm{e}}_z \,, \tag{3}$$

and

$$\Phi(\vec{r}_b) = ?\,, \qquad \vec{E}(\vec{r}_b) = ?\,, \qquad \vec{r}_b = (-1.3\,\mathrm{m})\,\hat{\mathrm{e}}_x + (-1.5\,\mathrm{m})\,\hat{\mathrm{e}}_y + (0.7\,\mathrm{m})\,\hat{\mathrm{e}}_z \,. \tag{4}$$

Give numerical results for all three vector components of the electric fields at $\vec{r}_a$ and $\vec{r}_b$. Express your results in SI mksA units, preferably, to an accuracy of at least 4 decimals! The vacuum permittivity is $\epsilon_0 = 8.8542 \times 10^{-12}\mathrm{C\,V^{-1}\,m^{-1}}$, so that $1/(4\pi\epsilon_0) = 8.9875 \times 10^9 \mathrm{N\,m^2/C^2}$.

(c.) Finally, calculate

$$\Phi_{\mathrm{diff}} = \Phi(\vec{r}_b) - \Phi(\vec{r}_a) \,. \tag{5}$$

Express your results in SI mksA units, preferably, to an accuracy of at least 4 decimals!

Task 2 (60 points)
Write a short essay to develop the concepts of a self-energy of an electrostatic field of a charge distribution, and the interaction energy of an electrostatic field of two charge distributions. Show, by an explicit calculation, the formula

$$W = 2W_0 + W_{\mathrm{int}} = 2 \times \frac{q^2}{8\pi\epsilon_0 a} - \frac{q^2}{4\pi\epsilon_0 R} > 0 = \frac{q^2}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{R} \right) \,, \tag{6}$$

for the total energy stored in the electrostatic field of a configuration consisting of two uniformly charged spheres, each of radius a, of charges $+q$ and $-q$, a distance R apart. You may use lecture notes. Show all your work! Write an account of the corresponding derivation, using explicit integrations in three dimensions, as given in the lecture notes, in your own words!

Task 3 (30 points)
Give an expression for the total field energy (sum of self energies and interaction energies) of the electrostatic field of three (!) uniformly charged spheres, each of charge $-q$ and radius a, with centers of the spheres at positions $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$.

The tasks are due Tuesday, 01-APR-2025. Have fun doing the problems!