

Task 1 (20 points)

Show that for a scalar field $\Psi = \Psi(\vec{r})$,

$$\oint_{\partial A} \Psi(\vec{s}) \, d\vec{s} = \int_A d\vec{A} \times \vec{\nabla} \Psi(\vec{r}) = \int_A \hat{n} \times \vec{\nabla} \Psi(\vec{r}) \, dA. \quad (1)$$

The latter two expressions are equal by definition, the first equality is to be shown. The method of proof is to apply Stokes's theorem to a vector field $\vec{V}(\vec{s}) = \vec{b} \Psi(\vec{s})$, where \vec{b} is an arbitrary constant vector. Please use this method.

Task 2 (30 points)

Define

$$||\vec{r}'|| = \sqrt{x^2 + y^2 + z^2}, \quad x_1 = x, \quad x_2 = y, \quad x_3 = z. \quad (2)$$

Show that for all $i, j = 1, 2, 3$, one has the relation

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \frac{1}{||\vec{r}'||} = \Theta(||\vec{r}'||) \left\{ \frac{3x_i x_j}{||\vec{r}'||^5} - \frac{\delta_{ij}}{||\vec{r}'||^3} \right\} - \frac{4\pi}{3} \delta_{ij} \delta^{(3)}(\vec{r}'). \quad (3)$$

Here, $\Theta(x)$ is the Heaviside step function, defined so that $\Theta(x) = 0$ for $x \leq 0$, and $\Theta(x) = 1$ for $x > 0$. Why is this result compatible with the defining equation for the Green function of the Poisson equation in three dimensions? Write a little essay on what happens when you let $i = j$ and sum over i from 1 to 3.

Task 3 (30 points)

Derive the wave equations

$$\left(\vec{\nabla}^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E} = \vec{0}, \quad \left(\vec{\nabla}^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{B} = \vec{0}, \quad (4)$$

for the electric and magnetic fields in a source-free region, $\rho = 0$ and $\vec{J} = 0$, by working in the SI mksA unit system in all intermediate steps.

Task 4 (20 points)

Take the divergence of both sides of the Ampere-Maxwell law,

$$\vec{\nabla} \times \vec{B}(\vec{r}, t) = \mu_0 \vec{J}(\vec{r}, t) + \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}(\vec{r}, t), \quad (5)$$

and show that you obtain the time derivative of Gauss's law.

The tasks are due Thursday, 06-MAR-2025. Have fun doing the problems!