Task 1 (50 points)

(a.) Define the concept of a Laurent series as opposed to a Taylor series. Define the concept of an Mth-order pole of a function of a complex variable, at a point $z = z_0$ in the complex plane. Show that the residue at an Mth-order pole can be calculated as follows,

$$\hat{\underset{z=z_0}{\mathrm{Res}}}\ f(z) = \frac{1}{(M-1)!} \left[\frac{\mathrm{d}^{M-1}\left\{ (z-z_0)^M\ f(z) \right\}}{\mathrm{d}z^{M-1}} \right]_{z=z_0} . \tag{1}$$

(b.) Additional task: Show that the above formula is consistent with what you expect for a first-order pole. In particular, **based on Eq.** (1), calculate the residues

$$R_1 = \hat{\underset{z=2\mathrm{i}}{\mathrm{Res}}}\ \frac{1}{(x^2+4)^2}, \qquad R_2 = \hat{\underset{z=2\mathrm{i}}{\mathrm{Res}}}\ \frac{1}{(x^2+4)^3}. \tag{2}$$

Show every step of your considerations accurately, and include all intermediate considerations.

Hint: Consult written distributed lecture notes.

Task 2 (40 points)

Calculate the following residues,

$$T_1 = \hat{\underset{z=0}{\mathrm{Res}}} \left(\frac{1}{z^3} \exp(-z^3)\ \sin(z) \right), \qquad T_2 = \hat{\underset{z=0}{\mathrm{Res}}} \left(\frac{1}{z^3} \exp(-z^2) \right), \tag{3}$$

$$T_3 = \hat{\underset{z=0}{\mathrm{Res}}} \left(\frac{1}{z^4} \exp(z)\ \cos(z) \right), \qquad T_4 = \hat{\underset{z=0}{\mathrm{Res}}} \left(\frac{1}{[\sin(z)]^2} \right). \tag{4}$$

Task 3 (40 points)

(a.) Calculate the integral

$$I = \int_{-\infty}^{\infty} \mathrm{d}x\ \frac{1}{(x^2+4)^2} = \frac{\pi}{16} \tag{5}$$

by residue calculus, mapping the integral onto a closed contour in the complex plane, and closing the contour, either in the upper or lower half of the complex plane. Any other way of analytic integration will result in zero points. **(b.)** Calculate the same integral by numerical integration, using your favorite computer system, and compare the result to $\pi/16 \approx 0.19635\ldots$.

Task 4 (50 points)

Calculate the integral

$$\int_{0}^{\infty} \mathrm{d}x\ \frac{\sin(x)}{x} \tag{6}$$

by residue calculus. (Any other way of calculation will result in zero points!) Hint: Symmetrize the integrand, write the sin function in terms of exponentials, displace the integration slightly above or below the real axis, and close the integration contour along a circle of infinite radius.

The tasks are due Thursday, 13-FEB-2025. Have fun doing the problems!