Task 1 (40 points). Calculate the closed-contour integrals

$$J_i = \oint_{C_i} \frac{1}{z}\,\mathrm{d}z = \oint_{C_i} \frac{1}{z(t)}\frac{\mathrm{d}z(t)}{\mathrm{d}t}\,\mathrm{d}t\,, \tag{1}$$

where $z \in \mathbb{C}$ is a complex variable, and the closed contours C_i with $i = 1, 2, 3$ are as follows:

- C_1 is a closed circular contour encircling the origin in the mathematically positive sense, with radius $R = |z| = 2$. Use the parameterization
  $C_1 = \{z = z(t)|z(t) = 2\exp(\mathrm{i}\,t),\ 0 < t < 2\pi\}$.

- C_2 is a closed circular contour encircling the origin in the mathematically negative sense, with radius $R = |z| = 7$. Use the parameterization
  $C_2 = \{z = z(t)|z(t) = 7\exp(-2\mathrm{i}\,t),\ 0 < t < \pi\}$. Watch the integration limits.

- C_3 is a closed circular contour encircling the origin TWICE in the mathematically negative sense, with radius $R = |z| = 54$. Use the parameterization
  $C_3 = \{z = z(t)|z(t) = 54\exp(-4\mathrm{i}\,t),\ 0 < t < \pi\}$. Watch the integration limits.

Calculate the contour integrals using the explicit parameterizations $z = z(t)$ given above, where t is the "time variable" along the contour. Any other solution will result in ZERO points. Also, show EVERY INTERMEDIATE STEP. This is very important.

Task 2 (50 points). **(a)** Let $f = f(z)$ be a complex function of a complex variable z, and let us assume that it can be written as $f = f_1 + \mathrm{i}f_2$, where f_1 and f_2 are the real and imaginary parts, respectively. Furthermore, let $\vec{F}^* = f_1\,\hat{e}_x - f_2\,\hat{e}_y$ be the "complex conjugate vector field". Show that, for closed-contour integrals, one has the formula

$$\oint f(z)\mathrm{d}z = \oint (f_1 + \mathrm{i}f_2)\,(\mathrm{d}x + \mathrm{i}\mathrm{d}y)$$

$$= \int_{\partial A} \vec{F}^* \cdot \mathrm{d}\vec{\ell} + \mathrm{i}\int_{\partial A} \vec{F}^* \cdot \mathrm{d}\vec{\ell}_\perp$$

$$= \int_A (\vec{\nabla} \times \vec{F}^*)_z\,\mathrm{d}A + \mathrm{i}\int_A \vec{\nabla} \cdot \vec{F}^*\,\mathrm{d}A\,, \qquad \mathrm{d}\vec{\ell}_\perp = \mathrm{d}\vec{\ell} \times \hat{e}_z\,. \tag{2}$$

All symbols are defined as in the lecture. You may use your lecture notes, but you must fill in any missing intermediate steps. Show your work!

(b) Find the vector field $\vec{F}^*$ corresponding to the function $f(z) = 1/z$ and find the line integrals over $\vec{F}^*$ which correspond to the contour integrals J_i over $f(z)$ with $i = 1$ and $i = 2$ (not $i = 3$).

Task 3 (50 points). Show that the Green function of the two-dimensional Poisson equation is

$$G(x, y) = \frac{1}{2\pi} \ln\left(\frac{\sqrt{x^2 + y^2}}{a}\right) \qquad \vec{\nabla}^2 G(x, y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) G(x, y) = \delta^{(2)}(x, y) = \delta(x)\,\delta(y)\,. \tag{3}$$

This can be done as follows: First, show that $\nabla^2 G(x, y) = 0$ for any point where $G = G(x, y)$ is differentiable, i.e., for any point excluding the origin. Then, show that the Dirac-δ term is obtained when one applies the divergence theorem to an infinitesimal circle around the origin.

Furthermore, show that $\vec{\nabla}G(x, y)$ is proportional to $\vec{F}^*(x, y)$, where $\vec{F}^*(x, y)$ is the vector field corresponding to the complex function $1/z$, which was discussed in task 2. Find the proportionality coefficient!

The tasks are due 12PM (noon) on Monday, 10–FEB–2025, in the instructor's mailbox.