In [U.D.Jentschura and G.S.Adkins, *Quantum Electrodynamics: Atoms, Lasers and Gravity*, World Scientific, 2022], two representations of the so-called Schrödinger–Coulomb Green function are being discussed which are extremely useful for practical calculations. These are the so-called Sturmian decomposition (in coordinate space) as well as the so-called Schwinger representation (in momentum space). The Sturmian decomposition can be derived based on a generalization of the so-called concatenation approach to the calculation of the Green function, and the Schwinger approach relies heavily on complex variables.

Task 1 (points to be allocated).

Discuss the derivation of the Sturmian decomposition of the Schrödinger–Coulomb Green function. It can be found in Eqs. (4.125) and (4.126) of [U.D.Jentschura and G.S.Adkins, *Quantum Electrodynamics: Atoms, Lasers and Gravity*, World Scientific, 2022], and reads as follows:

$$G(\vec{r}_1, \vec{r}_2, \nu) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} g_\ell(r_1, r_2, \nu)\, Y_{\ell m}(\theta_1, \varphi_1)\, Y^*_{\ell m}(\theta_2, \varphi_2)\,. \tag{1}$$

Here, the radial component is

$$g_\ell(r_1, r_2, \nu) = 2\mu \left(\frac{2Z}{a_0 \nu}\right)^{2\ell+1} \mathrm{e}^{-Z \frac{r_1 + r_2}{a_0 \nu}} \, (r_1\, r_2)^\ell \sum_{k=0}^{\infty} \frac{k!\, L_k^{2\ell+1}\left(\frac{2Zr_1}{a_0\nu}\right) \, L_k^{2\ell+1}\left(\frac{2Zr_2}{a_0\nu}\right)}{(k + 2\ell + 1)!\, (k + \ell + 1 - \nu)}\,. \tag{2}$$

Look up all relevant definitions of the Laguerre polynomials etc., the nuclear charge number Z, as well as the reduced mass μ. Derive the Sturmian decomposition based on the approach outlined in Chap. 4 of the cited book.

Task 2 (points to be allocated).

Discuss the derivation of Schwinger's momentum representation, found in Eq. (4.259) of the cited book,

$$\begin{aligned} G(\vec{p}, \vec{p}\,') = {} & 4\pi m X^3 \left(\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\pi\nu}}{2\sin(\pi\nu)}\right) \\ & \times \int_C \mathrm{d}\rho\, \rho^{-\nu} \frac{\partial}{\partial\rho} \frac{(1-\rho^2)/\rho}{\left[X^2\, (\vec{p} - \vec{p}\,')^2 + \frac{(1-\rho)^2}{4\rho}\, (X^2 + \vec{p}^{\,2})\, (X^2 + \vec{p}\,'^2)\right]^2}\,. \end{aligned} \tag{3}$$

The energy argument of the Green function is parameterized as

$$\nu = \frac{Z\alpha m}{X} = \frac{Z\alpha m}{\sqrt{-2mE}}\,. \tag{4}$$

Look up all relevant definitions (including the definition of the contour C) in Chap. 4 of the cited book and derive Eq. (3).

(Copies of Chap. 4 can be provided.)

The tasks are due on Tuesday, 03–DEC–2024.