Task 1 (30 points)
Consider once more the integral given in Eq. (12.64) of the lecture,

$$\int_{S_\epsilon} \mathrm{d}V\, \vec{\nabla}^2 g(\vec{r}) = \int_{\partial S_\epsilon} \mathrm{d}\vec{A} \cdot \vec{\nabla} g(\vec{r}) = A \int \mathrm{d}\Omega\, \epsilon^2\, \hat{r} \cdot \left(-\frac{1}{r^2}\right) \hat{r}\bigg|_{r=\epsilon} = -A \int \mathrm{d}\Omega = -A\, 4\pi \overset{!}{=} 1\,. \tag{1}$$

The integral

$$\int_{\partial S_\epsilon} \mathrm{d}\vec{A} \cdot \vec{\nabla} g(\vec{r}) \tag{2}$$

is taken over the surface of a sphere of radius ϵ. The notation used in Eq. (12.64) is actually a bit short-handed. If we wish to perform the surface integral in Eq. (1) very explicitly, then we should consider the definition of a surface integral,

$$\int \mathrm{d}\vec{A} \cdot \vec{F} = \int \mathrm{d}\theta \int \mathrm{d}\varphi \left(\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \varphi}\right) \cdot \vec{F}(\vec{r}(\theta, \varphi))\,. \tag{3}$$

Use $\vec{F}(\vec{r}(\theta, \varphi)) = \vec{\nabla} g(\vec{r}(\theta, \varphi))$, and parameterize the surface of the sphere of radius ϵ (why is this a good parameterization?) as

$$\vec{r}(\theta, \varphi) = \epsilon\, (\hat{\mathrm{e}}_x \sin\theta \cos\varphi + \hat{\mathrm{e}}_y \sin\theta \sin\varphi + \hat{\mathrm{e}}_z \cos\theta)\,. \tag{4}$$

Then, perform the surface integral $\int_{\partial S_\epsilon} \mathrm{d}\vec{A} \cdot \vec{\nabla} g(\vec{r})$ explicitly according to its very definition. Use appropriate limits (which ones?) for the integrals over θ and φ.

Task 2 (30 points)
Show how to go from the defining equation of the (ordinary) Bessel function,

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2} + 1\right) J_m(\rho) = 0\,, \tag{5}$$

to the relation

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2} + k^2\right) J_m(k\rho) = 0\,. \tag{6}$$

Then, look up the recurrence relations, asymptotic relations, as well as other properties, of the Bessel functions, from internet sources and textbooks. Write a “cheat sheet” with relevant formulas. Then, plot a few Bessel functions with indices $m = 0, 1, 2, \dots$, and investigate the asymptotic behavior for small and large argument visually. The extent of the “cheat sheet” is, to a certain extent, your choice.

Task 3 (30 points)
Consider the relation

$$\int_0^\infty \mathrm{d}\rho\, \rho\, J_m(k\,\rho)\; J_m(k'\,\rho) = \frac{1}{k}\,\delta(k - k') \tag{7}$$

given in the lecture, for the case $k = k_1 = 1.2$, and $k' = k_2 = 4.7$. Since $k_1 \neq k_2$, the right-hand side vanishes. So, the integral on the left-hand side should also vanish. Investigate this problem numerically. The integrand is highly oscillatory, and the integral does not converge very fast for $\rho \to \infty$. Try to find a good numerical routine which suppresses the magnitude of the integral for $k_1 = 1.2$, and $k_2 = 4.7$, as much as possible. Please avoid the use of excessive computational resource, or, running time.

Task 4 (30 points)
In the lecture, we had derived the expansion

$$g(\vec{r}, \vec{r}\,') = -\frac{1}{4\pi} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\varphi - \varphi')} \int_0^\infty \mathrm{d}k\, \mathrm{e}^{-k(z_> - z_<)}\, J_m(k\rho)\; J_m(k\rho') \tag{8}$$

for the Green function of the three-dimensional Poisson equation. Go through the derivation one more time, and write it up in detail.

The tasks are due Thursday, 21–NOV–2024. Have fun doing the problems!