Task 1 (30 points)
Show that $g_F(x - x') = |x - x'|/2$ is a Green function of the one-dimensional Poisson equation,

$$g_F(x - x') = \frac{|x - x'|}{2}\,, \qquad \frac{\partial^2}{\partial x^2} g_F(x - x') = \delta(x - x')\,, \tag{1}$$

and that

$$g_R(x - x') = \Theta(x - x')\,(x - x') \tag{2}$$

also is a valid Green function of the one-dimensional Poisson equation,

$$\frac{\partial^2}{\partial x^2} g_R(x - x') = \delta(x - x')\,. \tag{3}$$

Also, show that $f(x - x') = g_R(x - x') - g_F(x - x')$ is a solution of the homogeneous equation. (What is the homogeneous equation?)

Task 2 (30 points)
Show by explicit differentiation with respect to x, y, z and ξ that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \ln\left(\frac{\sqrt{x^2 + y^2}}{a}\right) = 0\,, \tag{4}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \frac{1}{\sqrt{x^2 + y^2 + z^2}} = 0\,, \tag{5}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \xi^2}\right) \frac{1}{x^2 + y^2 + z^2 + \xi^2} = 0\,, \tag{6}$$

provided $x \neq 0$, $y \neq 0$, $z \neq 0$ and $\xi \neq 0$. How is the derived result related to the Green functions of the Poisson equation in two, three and four dimensions? (Hint: For the first and the second of the above three equations, you may use your lecture notes and an appropriate notation of your choice, e.g., $r \equiv ||\vec{r}|| = \sqrt{x^2 + y^2 + z^2}$.)

Task 3 (30 points)
With the use of Gauss's theorem (divergence theorem), determine the prefactors which convert the above expressions given in Eqs. (4), (5) and (6) to solutions of the Poisson equations in two, three and four dimensions,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) g(x, y) = \delta^{(2)}(x, y)\,, \tag{7}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) g(x, y, z) = \delta^{(3)}(x, y, z)\,, \tag{8}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \xi^2}\right) g(x, y, z, \xi) = \delta^{(4)}(x, y, z, \xi)\,. \tag{9}$$

(Hint: You should formulate the divergence theorem in such a way that it is amenable to a generalization to four dimension. How would you paramerize a unit sphere imbedded in three dimensions? How would you paramerize a unit sphere imbedded in four dimensions?)

Task 4 (30 points)
Calculate the Green function of the Poisson equation in three dimensions,

$$g(\vec{r} - \vec{r}') = -\frac{1}{4\pi|\vec{r} - \vec{r}'|} \tag{10}$$

by Fourier transforming to wave vector space, and backtransforming to position space.

Ways to the solutions different from those outlined in the task receive zero points.
The tasks are due Tuesday, 12–NOV–2024. Have fun doing the problems!