Task 1 (40 points). You may use lecture notes! Consider the entire derivation of the general expression for the divergence of a vector field, in general coordinates. Show that

$$\nabla_i A^i = \partial_i A^i + \frac{A^i}{\sqrt{\det g}} \partial_i \sqrt{\det g} = \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g} A^i \right) . \tag{1}$$

You are encouraged to use lecture notes, but it must be visible that you redid every step yourself. Include comments in your derivation!

Task 2 (40 points). You may use lecture notes! Show that, in orthogonal curvilinear coordinates, one has

$$A_{(i)} = \frac{A^i}{h_{(i)}} , \qquad \nabla_i A^i = \frac{1}{h_{(1)} h_{(2)} h_{(3)}} \partial_i \left(h_{(1)} h_{(2)} h_{(3)} A^i \right) = \frac{1}{h_{(1)} h_{(2)} h_{(3)}} \partial_i \left(\frac{h_{(1)} h_{(2)} h_{(3)}}{h_{(i)}} A_{(i)} \right) . \tag{2}$$

Here, the $A_{(i)}$ are the physical components, while the A^i are the covariant components. The $h_{(i)}$ are the structure functions of the metric.

Task 3 (40 points). You are given the scalar function Φ, which is defined as

$$\Phi(x, y, z) = x^2 + y^2 \; = \; r^2 \sin^2 \theta = \Phi(r, \theta, \varphi) , \tag{3}$$

with the usual identification of Cartesian and spherical coordinates.

(a) Evaluate the gradient of $\Phi(x, y, z)$ in Cartesian coordinates. Then, evaluate the divergence of the gradient (again, in Cartesian coordinates).

(b) Evaluate the gradient of $\Phi(r, \theta, \varphi)$ in spherical coordinates and express your result in terms of the physical basis vectors $\hat{e}_{(r)}$, $\hat{e}_{(\theta)}$, and $\hat{e}_{(\varphi)}$. Then, evaluate the divergence of the gradient (again, in spherical coordinates).

(c) Evaluate the Laplacian of $\Phi(x, y, z)$ in Cartesian coordinates. Then, compare your results to the one you obtained in task (a) for the divergence of the gradient.

(d) Evaluate the Laplacian of $\Phi(r, \theta, \varphi)$ in spherical coordinates. Then, compare your results to the one you obtained in task (b) for the divergence of the gradient.

(e) Compare the results obtained in tasks (c) and (d) and show that they are equal.

The tasks are due on Tuesday, 05–NOV–2024.