Task 1 (40 points). You may use lecture notes! Start from the invariance of the full physical vector under coordinate transformations,

$$\vec{A} = A^i\,\vec{e}_i = \widetilde{A}^i\,\widetilde{\vec{e}}_i\,, \tag{1}$$

and using the quotient rule as discussed in the lecture, show that the covariant derivative

$$\nabla_j A^i \equiv \frac{\partial A^i}{\partial x^j} + \Gamma^i_{jk}\,A^k\,, \qquad \Gamma^i_{jk} = \frac{\partial \vec{e}_j}{\partial x^k}\cdot \vec{e}^{\,i} \tag{2}$$

is a mixed covariant-contravarinat tensor of the second rank. Furthermore, show the following identity for the Christoffel symbols of the second kind,

$$\Gamma^i_{jk} = \Gamma^i_{kj}\,. \tag{3}$$

Task 2 (40 points). You may use lecture notes! Show that the Christoffel symbols of the first kind,

$$\Gamma_{ijk} = \frac{\partial \vec{e}_i}{\partial x^j}\cdot \vec{e}_k\,, \tag{4}$$

are symmetric in the first two indices, *i.e.*, that

$$\Gamma_{ijk} = \Gamma_{jik}\,. \tag{5}$$

Then, starting from the definition given in Eq. (4), show that the Christoffel symbols of the first kind can be written as

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)\,. \tag{6}$$

Task 3 (40 points). You may use lecture notes! Show that, for spherical coordinates r, θ, and φ, with $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, defined (in the usual way) so that

$$\vec{r} = r\,\sin\theta\,\cos\varphi\,\hat{e}_x + r\,\sin\theta\,\sin\varphi\,\hat{e}_y + r\,\cos\theta\,\hat{e}_z\,, \tag{7}$$

one has the following Christoffel symbols of the first kind,

$$\Gamma_{221} = -r = -\Gamma_{122} = -\Gamma_{212}\,, \tag{8a}$$
$$\Gamma_{331} = -r\sin^2\theta = -\Gamma_{133} = -\Gamma_{313}\,, \tag{8b}$$
$$\Gamma_{332} = -\tfrac{1}{2}r^2\sin(2\theta) = -\Gamma_{233} = -\Gamma_{323}\,. \tag{8c}$$

Furthermore, show that the Christoffel symbols of the second kind are given as follows,

$$\Gamma^1_{22} = -r\,, \tag{9a}$$
$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = 1/r\,, \tag{9b}$$
$$\Gamma^2_{33} = -\tfrac{1}{2}\sin(2\theta)\,, \tag{9c}$$
$$\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta\,. \tag{9d}$$

The tasks are due on Thursday, 31–OCT–2024.