Task 1 (40 points). You may use lecture notes! Consider spherical coordinates, with the position vector $\vec{r}$ being written as follows,

$$\vec{r} = r\,\sin\theta\,\cos\varphi\,\hat{e}_x + r\,\sin\theta\,\sin\varphi\,\hat{e}_y + r\,\cos\theta\,\hat{e}_z\,. \tag{1}$$

Calculate the physical (unit) basis vectors

$$\hat{e}_r = \left(\left|\frac{\partial\vec{r}}{\partial r}\right|\right)^{-1}\frac{\partial\vec{r}}{\partial r}\,, \qquad \hat{e}_\theta = \left(\left|\frac{\partial\vec{r}}{\partial\theta}\right|\right)^{-1}\frac{\partial\vec{r}}{\partial\theta}\,, \qquad \hat{e}_\varphi = \left(\left|\frac{\partial\vec{r}}{\partial\varphi}\right|\right)^{-1}\frac{\partial\vec{r}}{\partial\varphi}\,. \tag{2}$$

Now consider a time-dependent position vector,

$$\vec{r}(t) = r(t)\,\sin\theta(t)\,\cos\varphi(t)\,\hat{e}_x + r(t)\,\sin\theta(t)\,\sin\varphi(t)\,\hat{e}_y + r(t)\,\cos\theta(t)\,\hat{e}_z\,. \tag{3}$$

Show that, with the common notation $\dot{f}(t) \equiv \mathrm{d}f(t)/\mathrm{d}t$, one has

$$\dot{\vec{r}}(t) = \dot{r}(t)\,\hat{e}_{(r)} + \dot{\theta}(t)\,r(t)\,\hat{e}_{(\theta)} + \dot{\varphi}(t)\,r(t)\,\sin\theta(t)\,\hat{e}_{(\varphi)}\,. \tag{4}$$

Task 2 (40 points). You may use lecture notes! Write the general transformation laws for the contravariant and covariant components of a vector. Show that, if one parameterizes space by three *general* coordinates x^1, x^2, and x^3, then the vectors

$$\vec{e}_i \equiv \frac{\partial\vec{r}}{\partial x^i}\,, \qquad i = 1, 2, 3, \tag{5}$$

transform with the covariant transformation law. Also, show that the components of the gradient vector,

$$(\vec{\nabla}f)_i \equiv \frac{\partial f}{\partial x^i}\,, \qquad i = 1, 2, 3, \tag{6}$$

transform as covariant components. For the spherical coordinates given above ($x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$), calculate the covariant vectors

$$\vec{e}_1 \equiv \vec{e}_r = \frac{\partial\vec{r}}{\partial r}\,, \qquad \vec{e}_2 \equiv \vec{e}_\theta = \frac{\partial\vec{r}}{\partial\theta}\,, \qquad \vec{e}_3 \equiv \vec{e}_\varphi = \frac{\partial\vec{r}}{\partial\varphi}\,. \tag{7}$$

Show that you can express the velocity in terms of contravariant vector components $\dot{x}^i$ and covariant basis vector $\vec{e}_i$ as

$$\dot{\vec{r}} = \dot{r}\,\vec{e}_r + \dot{\theta}\,\vec{e}_\theta + \dot{\varphi}\,\vec{e}_\varphi = \sum_{i=1}^{3}\dot{x}^i\,\vec{e}_i\,, \qquad \mathrm{d}\vec{r} = \mathrm{d}r\,\vec{e}_r + \vec{e}_\theta\,\mathrm{d}\theta + \vec{e}_\varphi\,\mathrm{d}\varphi\,. \tag{8}$$

Show that the latter form is obtained by multiplying the former by a infinitesimal time interval $\mathrm{d}t$. Watch the difference between the $\hat{e}_i$ and the $\vec{e}_i$!!!

Task 3 (40 points). You may use lecture notes! Calculate the modulus of the displacement

$$\mathrm{d}\vec{r}^{\,2} = (\mathrm{d}r\,\vec{e}_r + \vec{e}_\theta\,\mathrm{d}\theta + \vec{e}_\varphi\,\mathrm{d}\varphi)^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,, \tag{9}$$

with $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, and identify the components of the metric tensor

$$g_{ij} = ?\,, \qquad i, j = 1, 2, 3\,. \tag{10}$$

Show that, in the general case, the components of g_{ij} transform as the components of a second-rank, covariant tensor. Show also that, if the A_i are covariant vector components, then the components A^i, defined as

$$A^i = \sum_{i=1}^{3} g^{ij}\,A_j\,, \tag{11}$$

transform as contravariant components.

The tasks are due on Thursday, 24–OCT–2024.