Task 1 (50 points). **(a)** Show that $w = -u_\infty^* \, z$ with $u_\infty = u_{\infty,x} + \mathrm{i}\, u_{\infty,y}$ and $u_\infty^* = u_{\infty,x} - \mathrm{i}\, u_{\infty,y}$ is a suitable complex potential for uniform flow in the direction $-\vec{u}_\infty = -(u_{\infty,x}\, \hat{\mathrm{e}}_x + u_{\infty,y}\, \hat{\mathrm{e}}_z)$. In particular, find the velocity potential ϕ and the stream function ψ. **(c)** Make an approximate drawing showing the velocity field, and the stream lines, for the case $u_{\infty,x} = 3\mathrm{m/s}$ and $u_{\infty,y} = 1\mathrm{m/s}$ (the relative magnitude of $u_{\infty,x}/u_{\infty,x}$ is important for the drawing, but you can scale the x and y axes according to your taste). You may use computer-based plotting software, if so desired.

Task 2 (60 points). Investigate the complex potential

$$w = -\mathrm{i}\frac{\Gamma}{2\pi} \, \ln\left(\frac{z}{a}\right) , \tag{1}$$

where $z = x + \mathrm{i}y$. Here, a is the radius of the cylinder. **(a)** Show that the cylinder wall is a streamline. **(b)** Find the derivatives $\partial w/\partial z$, $\partial w/\partial x$, and $-\mathrm{i}\partial w/\partial y$, and show that they are all equal. **(c)** Find the corresponding velocity field and make a drawing for the case $a = 1\mathrm{m}$ and $\Gamma = 1\,\mathrm{m}^2/\mathrm{s}$. **(d)** Find the dependence of the modulus of the velocity field, $|\vec{u}|$, on the distance $r = \sqrt{x^2+y^2}$ from the center of the cylinder.

Task 3 (60 points). **(a)** Consider the complex potential

$$w = w(z) = -u_\infty \left(z + \frac{a^2}{z}\right) , \qquad z = x + \mathrm{i}y \,. \tag{2}$$

Decompose w into real and imaginary parts, i.e., calculate the velocity potential ϕ and the stream function ψ, as in the formula $w = \phi + \mathrm{i}\,\psi$, where $\phi = \mathrm{Re}\, w$ and $\psi = \mathrm{Im}\, w$. Show that

$$\phi = -u_\infty \left(x + \frac{a^2}{r^2} x\right) , \qquad \psi = -u_\infty \left(y - \frac{a^2}{r^2} y\right) . \tag{3}$$

(b) Show that the stream function $\psi(x, y)$ assumes a constant value (which one?) along the line $y = 0$ (trivial question) and also along the border of the circle, at $r^2 = a^2$, and interpret your result in terms of a streamline which extends along the y axis, and along the upper (or lower) part of the cylinder wall.

(c) Now, investigate the line of constant stream function $\psi = \psi(x, y) = -u_\infty\, c$ where c is a constant. Show that, for $x \to \pm\infty$, one has, asymptotically, $y = c + a^2\, c/x + \mathcal{O}(1/x^4)$ on the given streamline. Furthermore, show that the curve of constant stream function $\psi = \psi(x, y) = -u_\infty\, c$ meets the y axis at the point $x = 0$ (obviously!) and $y = \frac{1}{2}(c + \sqrt{4a^2 + c})$.

Task 4 (40 points). **(a)** Consider again the velocity potential for incompressible flow around a cylinder,

$$\phi(x, y) = -u_\infty \left(x + \frac{a^2}{r^2} x\right) , \qquad r = \sqrt{x^2+y^2} \geq a \,. \tag{4}$$

Show that, in the asymptotic limit, for a point $(x, y) = x\,\hat{e}_x + y\,\hat{e}_y$ far from the origin, one has

$$\vec{\nabla}\phi(x, y) \to -u_\infty\,\hat{e}_x \,, \qquad \sqrt{x^2+y^2} \to \infty. \tag{5}$$

and interpret your result geometrically.

(b) Evaluate the gradient of the velocity potential at $x = 0$ and $y = y$,

$$\vec{u}(0, y) = \vec{\nabla}\phi(0, y) \,, \qquad |y| > a \,, \tag{6}$$

and show that the maximum modulus of the velocity as the fluid moves around the cylinder in $2u_\infty$, i.e., twice the speed encountered at infinity.

Hint: You *might* obtain the vector-valued result $\vec{u}(0, y) = -u_\infty \left(1 + \frac{a^2}{y^2}\right)\hat{e}_x$, but this needs to be checked.

Show all your work!

The tasks are due on Thursday, 26–SEP–2024, with a possible extension to 27–SEP–2024.