Task 1 (40 points)
Consider, as in the lecture, the Pauli matrices,

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{1}$$

(a) Show that all three Pauli matrices are Hermitian (what is this?).

(b) Show that all three Pauli matrices have eigenvalues ± 1.

(c) Show that the Pauli matrices fulfill the relation $[\sigma_i, \sigma_j] = 2\mathrm{i} \sum_{k=1}^{3} \epsilon_{ijk} \sigma_k$, where the Levi–Cività tensor is denoted as ϵ_{ijk} and the commutator is denoted as $[.,.]$.

(d) Show that the Pauli matrices fulfill the relation $\{\sigma_i, \sigma_j\} = 2\mathrm{i} \delta_{i,j} \mathbb{1}_{2\times 2}$, where the Kronecker–$\delta$ is denoted as ϵ_{ijk} and the anti-commutator is denoted as $\{.,.\}$.

Task 2 (40 points)
Consider two quaternions q_1 and q_2, which have a real part and three imaginary parts, as given in the lecture,

$$q_1 = a_1 \mathbb{1}_{2\times 2} + b_1 \check{\mathrm{i}} + c_1 \check{\mathrm{j}} + d_1 \check{\mathrm{k}} = \begin{pmatrix} a_1 + \mathrm{i} b_1 & c_1 + \mathrm{i} d_1 \\ -c_1 + \mathrm{i} d_1 & a_1 - \mathrm{i} b_1 \end{pmatrix} , \tag{2}$$

$$q_2 = a_2 \mathbb{1}_{2\times 2} + b_2 \check{\mathrm{i}} + c_2 \check{\mathrm{j}} + d_2 \check{\mathrm{k}} = \begin{pmatrix} a_2 + \mathrm{i} b_2 & c_2 + \mathrm{i} d_2 \\ -c_2 + \mathrm{i} d_2 & a_2 - \mathrm{i} b_2 \end{pmatrix} . \tag{3}$$

The three imaginary units are given as follows,

$$\check{\mathrm{i}} = \mathrm{i}\, \sigma_z = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} , \qquad \check{\mathrm{j}} = \mathrm{i}\, \sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad \check{\mathrm{k}} = \mathrm{i}\, \sigma_x = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} . \tag{4}$$

(a) Read the lecture notes. Verify the concept of the "conjugate" of a quaternion. Use a suitable definition of the modulus of a quaternion and calculate $|q|^2 = q \cdot q^*$. Write a short essay.

(b) Verify, by an explicit calculation of the matrix product, the multiplication operation for quaternions:

$$q_3 = q_1 \cdot q_2 = \begin{pmatrix} a_3 + \mathrm{i} b_3 & c_3 + \mathrm{i} d_3 \\ -c_3 + \mathrm{i} d_3 & a_3 - \mathrm{i} b_3 \end{pmatrix} , \quad a_3 = a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 ,$$

$$b_3 = a_2 b_1 + a_1 b_2 - c_2 d_1 + c_1 d_2 , \quad c_3 = a_2 c_1 + a_1 c_2 + b_2 d_1 - b_1 d_2 , \quad d_3 = b_1 c_2 - b_2 c_1 + a_2 d_1 + a_1 d_2 . \tag{5}$$

(c) Show that the "commutator" $q_1 \cdot q_2 - q_2 - q_1$ fulfills

$$q_1 \cdot q_2 - q_2 \cdot q_1 = -2\mathrm{i} \sum_{ijk} \epsilon_{ijk} \sigma_i \, v_j \, w_k = -2\mathrm{i}\, \vec{\sigma} \cdot (\vec{v} \times \vec{w}) , \tag{6}$$

where $\vec{v} = (d_1, c_1, b_1)^{\mathrm{T}}$ is a column vector, and $\vec{w} = (d_2, c_2, b_2)^{\mathrm{T}}$ is the corresponding column vector for the second quaternion.

(d) Find a matrix representation for the inverse of a quaternion, e.g., in terms of its "conjugate" and modulus. Be inspired by the relation $z^{-1} = z^*/|z|^2$ for complex numbers.

Task 3 (20 extra points)
Read up on the complex contour representation of the Gamma function and write an essay about it.

The tasks are due Thursday, 19–SEP–2024.