Task 1 (40 points)
Derive the first three coefficients A, B, C in the Laurent expansion of $1/(z^4 \sin(z)\cos(z))$ about the origin,

$$f(z) = \frac{1}{z^4 \sin(z)\cos(z)} = \frac{A}{z^5} + \frac{B}{z^3} + \frac{C}{z} + \mathcal{O}(z)\,. \tag{1}$$

What does the symbol $\mathcal{O}(z)$ stand for? Consider the case $z = 0.05$. Show that, as you add more and more terms of the Laurent expansion, you get better approximations to the full function. Specifically, show that, with the term A/z^5 (coefficient A to be determined by you), you obtain an agreement with the full function up to a relative correction of 1.6×10^{-3}, which improves to 1.9×10^{-6} and 2.0×10^{-9} as you add more terms of the Laurent expansion.

Task 2 (40 points)
Calculate the following residues:

$$R_1 = \operatorname*{Res}_{z=0} \frac{1}{z^4\, \sin(z)\, \cos(z)}\,, \qquad R_2 = \operatorname*{Res}_{z=0} \frac{1}{z^2\, \sin(z)}\,, \tag{2}$$

$$R_3 = \operatorname*{Res}_{z=a} \frac{1}{z\,(a-z)\, \exp(z)}\,, \qquad R_4 = \operatorname*{Res}_{z=a} \frac{1}{z^3\,(a-z)^2\, \exp(z)}\,. \tag{3}$$

Here, a is a constant parameter. *Some of these tasks have a little wit to them. Please read carefully. Please take into account that not all residues are to be evaluated at $z = 0$.*

Task 3 (40 points)
(a) Consider, as in the lecture, the upper arc (half circle) of a circle centered at $R/2 + \mathrm{i}0$ and radius $R/2$ in the complex plane. Give a reason why this contour, traced in the mathematically positive sense (anti-clockwise), has the parameterization

$$C = \{z = z(t) = \tfrac{1}{2}R + \tfrac{1}{2}R\exp(\mathrm{i}t) \mid 0 < t < \pi\}\,, \tag{4}$$

while, if traced in the mathematically negative sense, it has the parameterization

$$C' = \{z = z(t) = \tfrac{1}{2}R - \tfrac{1}{2}R\exp(-\mathrm{i}t) \mid 0 < t < \pi\}\,. \tag{5}$$

Evaluate the contour integrals

$$I_1 = \int_C f(z)\,\mathrm{d}z\,, \qquad I_2 = \int_{C'} f(z)\,\mathrm{d}z\,, \qquad f(z) = z^2\,, \tag{6}$$

and show that $I_1 = -I_2$. Use the above parameterizations in your evaluations!

(b) Find the associated vector field $\vec{F}^*$ for $f(z) = z^2$. Find parameterizations for the associated paths P and P' in two-dimensional space, suitable for substitution in a line integral. Calculate the line integrals $J_1 = \int_P \vec{F}^* \cdot \mathrm{d}\vec{\ell}$ and $J_2 = \int_{P'} \vec{F}^* \cdot \mathrm{d}\vec{\ell}$ and show that $J_1 = I_1$, $J_2 = I_2$, by an explicit calculation.

(c) What happens when you trace the contours in task (a) "twice as fast", i.e., if you use the parameterizations

$$C = \{z = z(t) = \tfrac{1}{2}R + \tfrac{1}{2}R\exp(2\mathrm{i}t) \mid 0 < t < \pi/2\}\,, \tag{7}$$

$$C' = \{z = z(t) = \tfrac{1}{2}R - \tfrac{1}{2}R\exp(-2\mathrm{i}t) \mid 0 < t < \pi/2\}\,. \tag{8}$$

Give a general reason for your observations and do an explicit calculation, if possible!

Task 4 (20 points)
By a numerical integration, find the contour integral $\int_H f(z)\mathrm{d}z$ with $f(z) = 1/(1 + z^2/2)$, where H is the segment of the hyperbola described by the equation $y^2 - x^2 = 1$, extending from $x = -5$ to $x = +5$.

The tasks are due Thursday, 12–SEP–2024.