Task 1 (40 points). Show the identity (notation as in the lecture)

$$\int_{\partial A} \vec{F}(x,y) \cdot \mathrm{d}\vec{\ell}_\perp = \int_A \vec{\nabla} \cdot \vec{F}(x,y) \, \mathrm{d}A \,, \tag{1}$$

with reference to a small, but not necessarily infinitesimally small, reference area, namely, a rectangle with lower-left corner (x, y) and upper-right corner $(x + \delta x, y + \delta y)$. You may use your lecture notes for inspiration, but some explanatory remarks should be added to your derivation, as you see fits.

Task 2 (40 points). Show the identity (notation as in the lecture)

$$\oint_{\partial A} \vec{F}(x,y) \cdot \mathrm{d}\vec{\ell} = \int_A \left[\vec{\nabla} \times \vec{F} \right]_z \mathrm{d}A \tag{2}$$

with reference to a small, but not necessarily infinitesimally small, reference area, namely, a rectangle with lower-left corner (x, y) and upper-right corner $(x + \delta x, y + \delta y)$. *Does the contour ∂A encircle the area in the counterclockwise, or clockwise, direction? Please explain.* You may use your lecture notes for inspiration, but some explanatory remarks should be added to your derivation, as you see fits.

Task 3 (40 points). Calculate the complex closed-contour integral

$$I = \oint_C \frac{1}{z - z_0} \, \mathrm{d}z = \oint_C \frac{1}{z - \frac{1}{2} - \mathrm{i}\frac{1}{2}} \, \mathrm{d}z \,, \qquad z_0 = \frac{1}{2} + \frac{\mathrm{i}}{2} \,, \tag{3}$$

where C is the edge of a rectangle of side length unity, encircled counter-clockwise. So, we would have $C = z_1 \to z_2 \to z_3 \to z_4 \to z_1$, with $z_1 = 0$, $z_2 = 1$, $z_3 = 1 + \mathrm{i}$, and $z_4 = \mathrm{i}$. *Calculate the integral by explicitly evaluating all individual line integrals separately, as stated in the task. Just quoting or using a general result will result in zero credit.*

Task 4 (40 points). Discuss every step in the transformation of a closed complex contour integral and closed-path line integrals of an associated vector field,

$$\begin{aligned} \oint f(z) \mathrm{d}z &= \oint (f_1 + \mathrm{i} f_2)(\mathrm{d}x + \mathrm{i}\mathrm{d}y) = \oint (f_1 \mathrm{d}x - f_2 \mathrm{d}y) + \mathrm{i} \oint (f_2 \mathrm{d}x + f_1 \mathrm{d}y) \\ &= \int_{\partial A} \vec{F}^* \cdot \mathrm{d}\vec{\ell} + \mathrm{i} \int_{\partial A} \vec{F}^* \cdot \mathrm{d}\vec{\ell}_\perp \\ &= \int_A (\vec{\nabla} \times \vec{F}^*)_z \mathrm{d}A + \mathrm{i} \int_A \vec{\nabla} \cdot \vec{F}^* \mathrm{d}A \,. \end{aligned} \tag{4}$$

To this end, associate the complex function $f(z) = f_1(x,y) + \mathrm{i}\, f_2(x,y)$ where $z = x + \mathrm{i}y$, with the vector field $\vec{F}^*(x,y) = f_1(x,y)\, \hat{e}_x - f_2(x,y)\, \hat{e}_y$, as given in the lecture. *You may use lecture notes.*

Task 5 (40 points)
For a complex function

$$f(z) = f_1(x,y) + \mathrm{i}\, f_2(x,y) \,, \qquad z = x + \mathrm{i}y \,, \tag{5}$$

we have derived the Cauchy–Riemann differential equations

$$\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y} = 0 \,, \qquad \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x} = 0 \,. \tag{6}$$

Now consider a function $f = f(z^*)$ as opposed to $f = f(z)$. *Derive the analogue of the Cauchy–Riemann differential equations for a function $f = f(z^*)$.*

The tasks are due Thursday, 05–SEP–2024.