Task 1 (40 points)
Consider the rotation matrix $\mathbb{R}(\theta)$ in two dimension and the projector matrix $\mathbb{P}_x$,

$$\mathbb{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \mathbb{P}_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1}$$

Calculate the commutator

$$[\mathbb{R}(\theta), \mathbb{P}_x] = \mathbb{R}(\theta) \cdot \mathbb{P}_x - \mathbb{P}_x \cdot \mathbb{R}(\theta), \tag{2}$$

and interpret the result geometrically.

Task 2 (40 points)
Consider the matrix representation of a complex number $z = x + \mathrm{i}y$,

$$\mathbb{M}(z) = \begin{pmatrix} \mathrm{Re}\, z & -\mathrm{Im} z \\ \mathrm{Im}\, z & \mathrm{Re} z \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}. \tag{3}$$

Calculate $\mathbb{M}(z^{-1})$, i.e., the matrix representation of the inverse of a complex number, by expression $z^{-1} = z^*/|z|^2$ in terms of x and y, and using Eq. (3). Then, show that

$$\mathbb{M}(z^{-1}) = [\mathbb{M}(z)]^{-1}, \tag{4}$$

i.e., that the matrix representation of the inverse of a complex number z is equal to the inverse matrix of the matrix representation of z.

Task 3 (40 points)
Write a computer symbolic program, in your favorite system, which calculates the matrix product

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 7 \\ -3 & 1 \end{pmatrix} = ? \tag{5}$$

and provide a print-out of your program listing. You may use `python` or `julia` or any other programming language of your choice.

Task 4 (30 points)
Consider the function $f(z) = f(x + \mathrm{i}y) = \mathrm{e}^z \cos^2(z) = f_1(x, y) + \mathrm{i} f_2(x, y)$. Write the real and imaginary parts of $f(z)$ (i.e., the functions f_1 and f_2) as a function of x and y. Show that

$$\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y} = 0, \qquad \frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y} = 0. \tag{6}$$

Interpret your results in terms of the divergence of the vector fields $f_1\, \hat{\mathrm{e}}_x - f_2 \hat{\mathrm{e}}_y$ and $f_2\, \hat{\mathrm{e}}_x + f_1 \hat{\mathrm{e}}_y$.

Task 5 [*] (10 extra points)
Challenge: Express $\cos(3\theta)$ in terms of $\cos\theta$ and $\sin\theta$, your derivation being based on complex numbers.

The tasks are due Thursday, 29–AUG–2024.