Task 1 (30 points)
Give a heuristic argument for the validity of the following relations,

$$\int_{-1}^{1} \mathrm{d}u\, \delta(u-1) = \frac{1}{2}\,, \qquad \int_{-1}^{1} \mathrm{d}u\, u\, \delta(u-1) = \frac{1}{2}\,, \qquad \int_{-1}^{1} \mathrm{d}u\, u^3\, \delta(u-1) = \frac{1}{2}\,, \tag{1}$$

as well as

$$\int_{-1}^{1} \mathrm{d}u\, \delta(u+1) = \frac{1}{2}\,, \qquad \int_{-1}^{1} \mathrm{d}u\, u\, \delta(u+1) = -\frac{1}{2}\,, \qquad \int_{-1}^{1} \mathrm{d}u\, u^3\, \delta(u+1) = -\frac{1}{2}\,. \tag{2}$$

Task 3 (30 points)
Show that the charge distribution corresponding to a point charge q_0 located at Cartesian coordinates $(0, 0, a/2)$ and a point charge $-q_0$ located at Cartesian coordinates $(0, 0, -a/2)$ is given as follows (in spherical coordinates),

$$\rho(\vec{r}) = \frac{q_0}{(a/2)^2}\, \delta(r - a/2)\, [2\delta(\cos(\theta) - 1)]\, \delta(\varphi - \varphi_0) - \frac{q_0}{(a/2)^2}\, \delta(r - a/2)\, [2\delta(\cos(\theta) + 1)]\, \delta(\varphi - \varphi_0)\,. \tag{3}$$

Hint: Show that the volume integral of the first term is equal to q_0 (while the volume integral of the second term is $-q_0$). Then, investigate at which point the Dirac-δ functions peak, and show that the value of φ_0 does not matter.

Task 4 (30 points)
Calculate the multipole moments $q_{(\ell=1)(m=0)}$ and $q_{(\ell=3)(m=0)}$ of the charge distribution given in Eq. (3) **using the defining equation** (any other way results in zero credit!)

$$q_{10} = \int \rho\,(\vec{r})\;\, r^\ell\; Y^*_{10}\,(\theta, \varphi)\;\, \mathrm{d}^3 r\,, \qquad q_{30} = \int \rho\,(\vec{r})\;\, r^\ell\; Y^*_{30}\,(\theta, \varphi)\;\, \mathrm{d}^3 r\,. \tag{4}$$

Then, show that the result for the potential obtained by summing the relevant terms in the formula

$$\Phi\,(\vec{r}) = \frac{1}{\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{q_{\ell m}}{2\ell + 1}\, \frac{Y_{\ell m}\,(\theta, \varphi)}{r^{\ell+1}}\,. \tag{5}$$

is equivalent to the results communicated in task 4 of exam #2.

Hint: Use the results of task 1 above on this worksheet, i.e., the result communicated in Eqs. (1) and (2).

The tasks are due Tuesday, 30–APR–2024.