
Task 1 (30 points)
Verify the relation

$$\int Y^*_{\ell' \, m'}(\theta,\phi) \, Y_{\ell m}(\theta,\phi) \, \mathrm{d}\Omega = \int_0^{2\pi} \int_0^{\pi} Y^*_{\ell' \, m'}(\theta,\phi) \, Y_{\ell m}(\theta,\phi) \, \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = \delta_{\ell\ell'} \, \delta_{mm'} \,, \tag{1}$$

by an explicit calculation for all spherical harmonics with $\ell = \ell' = 1$ (nine pairs of ℓ and ℓ' are relevant). You may use considerations relevant for the averaging of coordinates over the unit sphere, or, an explicit representation of the integration over the solid angle.

Task 2 (20 points)
Verify the relation

$$\int Y^*_{20}(\theta,\phi) \, Y_{20}(\theta,\phi) \, \mathrm{d}\Omega = 1 \qquad \text{(no summation over } m!) \tag{2}$$

by an explicit calculation for the spherical harmonic with $\ell = 2$, and $m = 0$.

Task 3 (80 points)
In the lecture, we had derived the relation

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y_{\ell m}(\theta,\varphi) \, Y^*_{\ell m}(\theta',\varphi') \,, \tag{3}$$

as the multipole expansion of the Green function of the three-dimensional Possion equation. However, we had used a slightly different normalization in the lecture; namely, we had considered the potential generated by a point charge as opposed to the expression $1/|\vec{r} - \vec{r}'|$.

Write an essay about the precise derivation of this multipole expansion, in the normalization given in Eq. (3). Preferably, outline every single intermediate step and convince yourself that Eq. (3) remains correct in asymptotic limits (in particular, in the limit $r \to \infty$ while keeping r' constant).

Task 3 (20 EXTRA!!! points)
What is the parity transformation? How is the parity transformation implemented for spherical harmonics? What is the transformation property of spherical harmonics under parity?

The tasks are due Thursday, 18–APR–2024.