Task 1 (50 points+20 extra points) (Variational Calculus) Write a short essay on the definition of the functional derivative of a function. Consider the energy functional

$$E[\Phi] = \frac{\epsilon_0}{2} \int \mathrm{d}^3 r \, \left| \vec{\nabla} \Phi(\vec{r}) \right|^2 \tag{1}$$

of an electrostatic potential $\Phi = \Phi(\vec{r})$. Show that

$$\frac{\delta E[\Phi]}{\delta \Phi(\vec{r})} = -\epsilon_0 \vec{\nabla}^2 \Phi(\vec{r}) \,. \tag{2}$$

Use either the definition of the functional derivative on the basis of an addition of a Dirac-δ function, or via the addition of the variation $\delta\Phi(\vec{r})$ to the argument function. If you use $\delta\Phi(\vec{r})$, then pay close attention to the boundary conditions imposed on the variation $\delta\Phi(\vec{r})$ when you do partial integrations. *Show and comment every step in your derivation.* Interpret the result in terms of the connection of the variational principle, the minimization of the electrostatic field energy, and the Laplace equation. *Extra points:* In which sense is the operator-valued second functional derivative [to be shown]

$$\frac{\delta^2 E[\Phi]}{\delta \Phi(\vec{r}\,') \, \delta \Phi(\vec{r})} = -\epsilon_0 \, \delta^{(3)}(\vec{r} - \vec{r}\,') \, \vec{\nabla}^2 \tag{3}$$

positive, i.e., in which sense can we say that

$$\text{“}\frac{\delta^2 E[\Phi]}{\delta \Phi(\vec{r}\,') \, \delta \Phi(\vec{r})} > 0\text{”} \, ? \tag{4}$$

Think about the “positivity of a matrix” and a connection of this concept to the positivity of all of its eigenvalues.

Task 2 (40 points) Reconsider the variational problem of the calculation of the potential in a *cylindrical* capacitor, with a more complex trial potential (with the same boundary conditions as in the lecture)

$$w(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \rho) = w(\rho) = \mathcal{A} + \mathcal{B}\,\rho + \mathcal{C}\,\rho^2 + \mathcal{D}\,\rho^3 \,, \qquad w(\rho = b) = 0 \,, \qquad w(\rho = c) = V_0 \,. \tag{5}$$

Use the parameters $b = 0.2\,\mathrm{cm}$ and $c = 0.8\,\mathrm{cm}$ and plot the variational solution you obtain. Convince yourself that you obtain an even better approximation to the analytic solution than in the lecture, where we had used a three-parameter variational *ansatz*.

The tasks are due Thursday, 09–MAY–2024. No extension of the deadline will be possible.