Task 1 (20 points) (Bessel Functions I) Verify the two asymptotic relations

$$J_m(x) \to \frac{x^2}{(2m)!!} \quad (x \to 0)\,, \quad J_m(x) \to \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{2}\left(m - \frac{1}{2}\right)\right) \quad (x \to \infty)\,, \tag{1}$$

by way of example, using graphical and numerical computer algebraic programs of your choice, choosing for m suitable integer values (perhaps, not too large). One chooses a domain of positive x, i.e., $x > 0$. Two plots will be made, for the regions $x \ll m$ and $x \gg m$.

Task 1 (20 points) (Bessel Functions II) Verify the recursion relations

$$J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x)\,, \tag{2a}$$

$$J_{m-1}(x) - J_{m+1}(x) = 2J'_m(x)\,, \tag{2b}$$

by way of example, using graphical and numerical computer algebraic programs of your choice, choosing for m suitable integer values (perhaps, not too large). One selects the regime $x > 0$, and, perhaps, $x \sim m$.

Task 3 (20 points) (Spherical Bessel Functions I) Verify the two asymptotic relations

$$j_\ell(x) \to \frac{x^\ell}{(2\ell+1)!!}\,, \qquad (x \to 0)\,, \qquad j_\ell(x) \to \frac{1}{x} \sin\left(x - \frac{\ell\pi}{2}\right)\,, \qquad (x \to \infty)\,, \tag{3}$$

by way of example, using graphical and numerical computer algebraic programs of your choice, choosing for ℓ suitable integer values (perhaps, not too large). One chooses a domain of positive x, i.e., $x > 0$. Two plots will be made, for the regions $x \ll \ell$ and $x \gg \ell$.

Task 4 (20 points) (Spherical Bessel Functions II) Verify the recursion relations

$$j_{\ell-1}(x) + j_{\ell+1}(x) = \frac{2\ell+1}{x}\, j_\ell(x), \tag{4a}$$

$$\ell\, j_{\ell-1}(x) - (\ell+1)\, j_{\ell+1}(x) = (2\ell+1)\, j_\ell(x), \tag{4b}$$

by way of example, using graphical and numerical computer algebraic programs of your choice, choosing for ℓ suitable integer values (perhaps, not too large). One chooses a domain of positive x, i.e., $x > 0$, and $x \sim \ell$.

Task 5 (20 points) (Laplace and Helmholtz Equations) Write an essay on the two questions: (i) How does the ordinary Bessel function enter the solution of the Laplace equation in cylindrical coordinates? (ii) How does the spherical Bessel function enter the solution of the Helmholtz equation in spherical coordinates?

The tasks are due on Thursday, 02–MAY–2024.