Physics/6111 Electrodynamics I Dr. Ulrich Jentschura Spring Semester 2024

Missouri S & T (Rolla, Missouri) Exercise Sheet 10 25–APR–2024

Task 1 (40 points) (Non–Uniform Convergence) Investigate the potential given in Eq. (6.75) of the lecture notes,

$$\Phi\left(x,y\right) = \frac{4\Phi_0}{\pi} \sum_{n=0}^{\infty} \frac{\sin\left[\left(2n+1\right)\pi x/a\right]}{2n+1} \, \frac{\sinh\left[\left(2n+1\right)\pi y/a\right]}{\sinh\left[\left(2n+1\right)\pi b/a\right]}\,, \tag{1}$$

Show the following properties which illustrate the non-uniform convergence at the point $x = 0$, and $y = b$,

$$\lim_{\epsilon\to 0^+} \Phi(0, b-\epsilon) = 0\,, \qquad \lim_{\epsilon\to 0^+} \Phi(\epsilon, b) = \Phi_0\,. \tag{2}$$

Task 2 (80 points) (Laplace Equation and Potential) Investigate the potential given in Eq. (6.90) of the lecture notes,

$$\Phi(x,y) = \frac{V_0}{\mathrm{i}\pi}\,\ln\left(\frac{\mathrm{e}^{\mathrm{i}\pi y/b} + \mathrm{e}^{\pi x/b}}{\mathrm{e}^{-\mathrm{i}\pi y/b} + \mathrm{e}^{\pi x/b}}\right)\,. \tag{3}$$

Show that, for real and positive $x > 0$, and $0 < y < b$, the result for $\Phi(x,y)$ is real rather than complex. Then, plot the potential given in Eq. (3), as a function of x and y, for your preferred choice of the parameter b, over a suitable range of x and y variables.

Task 3 (40 points) (Non–Uniform Convergence) Investigate, once more, Eq. (6.90) of the lecture notes, which is given above in Eq. (3). Show that

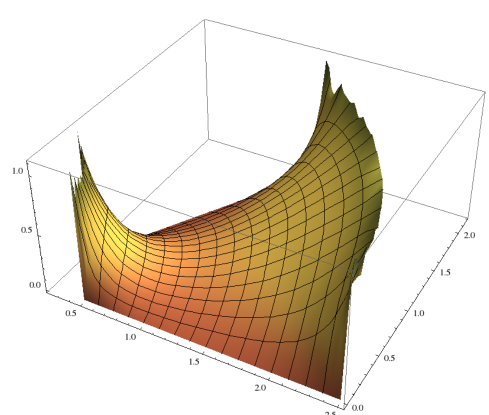
$$\lim_{\epsilon\to 0^+} \Phi(\epsilon, b) = 0\,, \qquad \lim_{\epsilon\to 0^+} \Phi(0, b-\epsilon) = V_0\,. \tag{4}$$

For the latter relation, use the result $\lim_{\eta\to 0^+} \ln(-1 \pm \mathrm{i}\eta) = \pm\mathrm{i}\pi$. **Hint:** You may want to show that, as you calculate $\lim_{\epsilon\to 0^+} \Phi(0, b-\epsilon)$, you obtain an expression where the imaginary part of the argument of the logarithm is positive.

Task 4 (60 points+20 extra points) (Laplace Equation) You are given a two-dimensional wedge-shaped region between $\theta = 0$ and $\theta = 60^\circ$, which extends in the radial distance from $\rho = |\vec{\rho}| = \rho_a$ to $\rho = |\vec{\rho}| = \rho_b$. This constitutes a "piece of cake" with a "bite taken from the inside". Furthermore, you are given the boundary conditions $\Phi(|\vec{\rho}| = a, \theta) = f_a(\theta) = \Phi_a = \Phi_0$, $\Phi(|\vec{\rho}| = b, \theta) = f_b(\theta) = \Phi_b = \Phi_0$, i.e., a constant potential on the inner and outer edge of the "piece of cake", as well as $\Phi(\rho, \theta = 0) = \Phi(\rho, \theta = \beta) = 0$. The latter condition implies a vanishing potential on the left and right stripes of the "piece of the cake". Solve the Laplace boundary-value problem

$$\vec{\nabla}^2 \Phi(\vec{\rho}) = 0\,, \tag{5}$$

using a suitable series expansion of the potential, as in Eqs. (6.123)—(6.128) of the lecture notes. *Extra points:* Plot the solution. For the parameters $\rho_a = 0.5$, $\rho_b = 2.5$, and $\Phi_0 = 0.95$, show that one obtains (using a suitable number of expansion coefficients) a plot similar to



The tasks are due on Thursday, 02–MAY–2024. No extension of the deadline will be possible.