Task 1 (60 points)
Consider the Poisson equation (in three dimensions) for a charge distribution composed of point charges,

$$\vec{\nabla}^2 \, \Phi(\vec{r}) = -\frac{1}{\epsilon_0} \, \rho(\vec{r}) \,, \qquad \rho(\vec{r}) = q_1 \, \delta^{(3)}(\vec{r} - \vec{r}_1) + q_2 \, \delta^{(3)}(\vec{r} - \vec{r}_2) \,. \tag{1}$$

Symbols are defined as in the lecture. The two point charges are places at $\vec{r}_1$ and $\vec{r}_2$.

(a.) Using the Green function of the three-dimensional Poisson equation, calculate the **electrostatic potential** $\Phi(\vec{r})$ and the electric field $\vec{E}(\vec{r})$,

$$\Phi(\vec{r}) = -\frac{1}{\epsilon_0} \int \mathrm{d}^3 r' \, g(\vec{r} - \vec{r}\,') \, \rho(\vec{r}\,') \,, \qquad \vec{E}(\vec{r}) = -\vec{\nabla} \Phi(\vec{r}) \,, \tag{2}$$

by finding an analytic expression involving the parameters q_1, q_2, $\vec{r}$, $\vec{r}_1$ and $\vec{r}_2$. **When calculating the expression $\int \mathrm{d}^3 r' \, g(\vec{r} - \vec{r}\,') \, \rho(\vec{r}\,')$, carefully distiguish independent arguments of functions, integration variables, and parameters!!!**

(b.) Assume $q_1 = 0.04\,\mathrm{C}$ and $q_2 = 0.16\,\mathrm{C}$, $\vec{r}_1 = \vec{0}$, and $\vec{r}_2 = (3.5\,\mathrm{m})\,\hat{\mathrm{e}}_x$. Calculate the quantities

$$\Phi(\vec{r}_a) = ?\,, \qquad \vec{E}(\vec{r}_a) = ?\,, \qquad \vec{r}_a = (0.3\,\mathrm{m})\,\hat{\mathrm{e}}_x + (1.5\,\mathrm{m})\,\hat{\mathrm{e}}_y + (0.1\,\mathrm{m})\,\hat{\mathrm{e}}_z \,, \tag{3}$$

and

$$\Phi(\vec{r}_b) = ?\,, \qquad \vec{E}(\vec{r}_b) = ?\,, \qquad \vec{r}_b = (-0.3\,\mathrm{m})\,\hat{\mathrm{e}}_x + (-1.5\,\mathrm{m})\,\hat{\mathrm{e}}_y + (0.7\,\mathrm{m})\,\hat{\mathrm{e}}_z \,. \tag{4}$$

Give numerical results for all three vector components of the electric fields at $\vec{r}_a$ and $\vec{r}_b$. Express your results in SI mksA units, preferably, to an accuracy of at least 4 decimals! The vacuum permittivity is $\epsilon_0 = 8.8542 \times 10^{-12} \mathrm{C\,V^{-1}\,m^{-1}}$, so that $1/(4\pi\epsilon_0) = 8.9875 \times 10^9 \mathrm{N\,m^2/C^2}$.

(c.) Finally, calculate

$$\Phi_{\mathrm{diff}} = \Phi(\vec{r}_b) - \Phi(\vec{r}_a) \,. \tag{5}$$

Express your results in SI mksA units, preferably, to an accuracy of at least 4 decimals!

Task 2 (60 points)
Write a short essay to develop the concepts of a self-energy of an electrostatic field of a charge distribution, and the interaction energy of an electrostatic field of two charge distributions. Show, by an explicit calculation, the formula

$$W = 2W_0 + W_{\mathrm{int}} = 2 \times \frac{q^2}{8\pi\epsilon_0 a} - \frac{q^2}{4\pi\epsilon_0 R} > 0 = \frac{q^2}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{R} \right) \,, \tag{6}$$

for the total energy stored in the electrostatic field of a configuration consisting of two uniformly charged spheres, each of radius a, of charges $+q$ and $-q$, a distance R apart. You may use lecture notes. Show all your work!

Task 3 (30 points)
Give an expression for the total field energy (sum of self energies and interaction energies) of the electrostatic field of three (!) uniformly charged spheres, each of charge $-q$ and radius a, with centers of the spheres at positions $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$.

Task 4 (10 points)
Write a "cheat sheet" with the most important formulas from the lecture, comprising all aspects since the start of the semester.

The tasks are due Thursday, 04-APR-2024. Have fun doing the problems!