Task 1 (20 points)
Show that for a scalar field $\Psi = \Psi(\vec{r})$,

$$\oint_{\partial A} \Psi(\vec{s})\, \mathrm{d}\vec{s} = \int_A \mathrm{d}\vec{A} \times \vec{\nabla}\Psi(\vec{r}) = \int_A \hat{n} \times \vec{\nabla}\Psi(\vec{r})\, \mathrm{d}A\,. \tag{1}$$

The latter two expressions are equal by definition, the first equality is to be shown. The method of proof is to apply Stokes's theorem to a vector field $\vec{V}(\vec{s}) = \vec{b}\,\Psi(\vec{s})$, where $\vec{b}$ is an arbitary constant vector. Please use this method.

Task 2 (30 points)
Define

$$||\vec{r}|| = \sqrt{x^2 + y^2 + z^2}\,, \qquad x_1 = x\,, \quad x_2 = y\,, \quad x_3 = z\,. \tag{2}$$

Show that for all $i, j = 1, 2, 3$, one has the relation

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{1}{||\vec{r}||} = \Theta(||\vec{r}||)\left\{\frac{3x_i x_j}{||\vec{r}||^5} - \frac{\delta_{ij}}{||\vec{r}||^3}\right\} - \frac{4\pi}{3}\delta_{ij}\,\delta^{(3)}(\vec{r})\,. \tag{3}$$

Here, $\Theta(x)$ is the Heaviside step function, defined so that $\Theta(x) = 0$ for $x \leq 0$, and $\Theta(x) = 1$ for $x > 0$. Why is this result compatible with the defining equation for the Green function of the Poisson equation in three dimensions? Write a little essay on what happens when you let $i = j$ and sum over i from 1 to 3.

Task 3 (30 points)
Derive the wave equations

$$\left(\vec{\nabla}^{\,2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \vec{0}\,, \qquad \left(\vec{\nabla}^{\,2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{B} = \vec{0}\,, \tag{4}$$

for the electric and magnetic fields in a source-free region, $\rho = 0$ and $\vec{J} = 0$, by working in the SI mksA unit system in all intermediate steps.

Task 4 (20 points)
Take the divergence of both sides of the Ampere–Maxwell law,

$$\vec{\nabla} \times \vec{B}(\vec{r}, t) = \mu_0\,\vec{J}(\vec{r}, t) + \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{r}, t)\,, \tag{5}$$

and show that you obtain the time derivative of Gauss's law.

Task 5 (40 points)
Solve the differential equation governing the static magnetic field generated by a current in the z direction,

$$\vec{\nabla} \times \vec{B}(\vec{r}) = \mu_0\, I_0\, \hat{\mathrm{e}}_z\, \delta(x)\, \delta(y)\,, \tag{6}$$

with the help of the two-dimensional Green function of the Poisson equation. Verify your answer by an explicit calculation! **Hint:** Try to draw a picture and visualize the current. **Hint:** Try an ansatz proportional to $\vec{B}(\vec{r}) = \hat{\mathrm{e}}_z \times \vec{\nabla}\ln(\rho/a)$, where $\rho = \sqrt{x^2 + y^2}$.

The tasks are due Thursday, 21-MAR-2024. Have fun doing the problems!