Task 1 (30 points)
Show by explicit differentiation that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\ln\left(\frac{\sqrt{x^2+y^2}}{a}\right)=0\,, \tag{1}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{1}{\sqrt{x^2+y^2+z^2}}=0\,, \tag{2}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial \xi^2}\right)\frac{1}{x^2+y^2+z^2+\xi^2}=0\,, \tag{3}$$

provided $x \neq 0$, $y \neq 0$, $z \neq 0$ and $\xi \neq 0$. How is the derived result related to the Green functions of the Poisson equation in two, three and four dimensions? (Hint: For the first and the second of the above three equations, you may use your lecture notes and an appropriate notation of your choice, e.g., $r \equiv ||\vec{r}|| = \sqrt{x^2+y^2+z^2}$.)

Task 2 (30 points)
With the use of Gauss's theorem (divergence theorem), determine the prefactors which lead to solutions of the Poisson equations in two, three and four dimensions,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)g(x,y)=\delta^{(2)}(x,y)\,, \tag{4}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)g(x,y,z)=\delta^{(3)}(x,y,z)\,, \tag{5}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial \xi^2}\right)g(x,y,z,\xi)=\delta^{(4)}(x,y,z,\xi)\,. \tag{6}$$

(Hint: You should formulate the divergence theorem in such a way that it is amenable to a generalization to four dimension. How would you paramerize a unit sphere imbedded in three dimensions? How would you paramerize a unit sphere imbedded in four dimensions?)

Task 3 (30 points)
Calculate the Green function of the Poisson equation in three dimensions,

$$g(\vec{r}-\vec{r}\,')=-\frac{1}{4\pi|\vec{r}-\vec{r}\,'|} \tag{7}$$

by Fourier transforming to wave vector space, and backtransforming to position space.

Task 4 (30 points)
Show that $g(x-x')$ is a Green function of the one-dimensional Poisson equation,

$$g(x-x')=\frac{|x-x'|}{2}\,, \qquad \frac{\partial^2}{\partial x^2}g(x-x')=\delta(x-x')\,, \tag{8}$$

and that

$$\tilde{g}(x-x')=\Theta(x-x')\,(x-x') \tag{9}$$

also is a valid Green function of the one-dimensional Poisson equation,

$$\frac{\partial^2}{\partial x^2}\tilde{g}(x-x')=\delta(x-x')\,. \tag{10}$$

Also, show that

$$f(x-x')=g(x-x')-\tilde{g}(x-x') \tag{11}$$

is a solution of the homogeneous equation.

The tasks are due Thursday, 07–MAR–2024. Have fun doing the problems!