Task 1 (50 points)
Consider Stokes's theorem in three dimension, formulated for a general three-dimensional vector field $\vec{F} = \vec{F}(\vec{r})$, and an infinitesimal line element $\mathrm{d}\vec{s}$, and area element $\mathrm{d}\vec{A}$,

$$\oint_{\partial A} \vec{F}(\vec{s}) \cdot \mathrm{d}\vec{s} = \oint_{\partial A} \vec{F}(\vec{s}(t)) \cdot \frac{\mathrm{d}\vec{s}(t)}{\mathrm{d}t} \, \mathrm{d}t = \int_A \left(\vec{\nabla} \times \vec{F}(\vec{r}) \right) \cdot \mathrm{d}\vec{A} \,, \tag{1}$$

where $\vec{s} = \vec{s}(t)$ parameterizes the boundary of the surface and

$$F(\vec{r}) = F_x(\vec{r}) \, \hat{\mathrm{e}}_x + F_y(\vec{r}) \, \hat{\mathrm{e}}_y + F_z(\vec{r}) \, \hat{\mathrm{e}}_z \,, \tag{2}$$

$$\mathrm{d}\vec{s} = \hat{\mathrm{e}}_x \, \mathrm{d}s_x + \hat{\mathrm{e}}_y \, \mathrm{d}s_y + \hat{\mathrm{e}}_z \, \mathrm{d}s_z \,, \tag{3}$$

$$\mathrm{d}\vec{A} = \hat{n} \, |\mathrm{d}\vec{A}| \,, \tag{4}$$

and $\hat{n}$ is the surface normal. In the lecture notes, the theorem is shown for the case of a z oriented surface normal, with reference to a small, but not infinitesimally small surface δA, i.e., when $\delta\vec{A} = \hat{n} \, \delta A$ is

- Case I: a rectangle, $\hat{n} = \hat{\mathrm{e}}_z$, with edges at x and $x + \delta x$, as well as y and $y + \delta y$ (Lecture, done, nothing to do).

One can then applied the "patchwork principle" to conclude that the theorem also holds for a large oriented surface A with boundary ∂A. Include a small write-up of the corresponding lecture notes. Now, show by an explicitly calculation that the theorem also holds when $\delta\vec{A} = \hat{n} \, \delta A$ is

- Case II: a rectangle, $\hat{n} = \hat{\mathrm{e}}_x$, with edges at y and $y + \delta y$, as well as z and $z + \delta z$,
- Case III: a rectangle, $\hat{n} = \hat{\mathrm{e}}_y$, with edges at x and $x + \delta x$, as well as z and $z + \delta z$,

Then, explain how the considerations change for

- Case IV: a rectangle, $\hat{n} = -\hat{\mathrm{e}}_x$, with edges at y and $y + \delta y$, as well as z and $z + \delta z$,
- Case V: a rectangle, $\hat{n} = -\hat{\mathrm{e}}_y$, with edges at x and $x + \delta x$, as well as z and $z + \delta z$,

Hint: How do you have to orient the line integral over $\mathrm{d}\vec{s}$ in each case?

Task 2 (30 points)
Complete the proof of Gauss's theorem for a general vector field $\vec{F} = \vec{F}(\vec{r})$,

$$\int_{\partial V} \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{A} = \int_V \vec{\nabla} \cdot \vec{F}(\vec{r}) \, \mathrm{d}V \,, \tag{5}$$

with reference to a small, but not infinitesimally small cubic reference volume δV, with edges at x and $x + \delta x$, y and $y + \delta y$, and z and $z + \delta z$. Consider all sides of the cubic reference volume δV and explain how the "Lego principle" completes the proof for a "macroscopic" reference volume V.

Task 3 (20 points)
Explain how the application of Stokes's and Gauss's theorem transform the Maxwell equations from integral to differential form.

The tasks are due Thursday, 22-FEB-2024. Have fun doing the problems!