**Task 1** (45 points). Calculate the closed-contour integrals

$$J_i = \oint_{C_i} \frac{1}{z}\, \mathrm{d}z = \oint_{C_i} \frac{1}{z(t)} \frac{\mathrm{d}z(t)}{\mathrm{d}t}\, \mathrm{d}t\,, \tag{1}$$

where $z \in \mathbb{C}$ is a complex variable, and the closed contours $C_i$ with $i = 1, 2, 3$ are as follows:

- $C_1$ is a closed circular contour encircling the origin in the mathematically positive sense, with radius $R = |z| = 2$. Use the parameterization $C_1 = \{z = z(t) | z(t) = 2\, \exp(\mathrm{i}\, t),\ 0 < t < 2\pi\}$.
- $C_2$ is a closed circular contour encircling the origin in the mathematically negative sense, with radius $R = |z| = 7$. Use the parameterization $C_2 = \{z = z(t) | z(t) = 7\, \exp(-2\mathrm{i}\, t),\ 0 < t < \pi\}$. Watch the integration limits.
- $C_3$ is a closed circular contour encircling the origin TWICE in the mathematically negative sense, with radius $R = |z| = 54$. Use the parameterization $C_3 = \{z = z(t) | z(t) = 54\, \exp(-4\mathrm{i}\, t),\ 0 < t < \pi\}$. Watch the integration limits.

Calculate the contour integrals using the explicit parameterizations $z = z(t)$ given above, where $t$ is the "time variable" along the contour. Any other solution will result in ZERO points. Also, show EVERY INTERMEDIATE STEP. This is very important.

**Task 2** (55 points). **(a)** Let $f = f(z)$ be a complex function of a complex variable $z$, and let us assume that it can be written as $f = f_1 + \mathrm{i} f_2$, where $f_1$ and $f_2$ are the real and imaginary parts, respectively. Furthermore, let $\vec{F}^* = f_1\, \hat{\mathrm{e}}_x - f_2\, \hat{\mathrm{e}}_y$ be the "complex conjugate vector field". Show that, for closed-contour integrals, one has the formula

$$\begin{aligned}
\oint f(z)\mathrm{d}z &= \oint (f_1 + \mathrm{i} f_2)\, (\mathrm{d}x + \mathrm{i}\mathrm{d}y) \\
&= \int_{\partial A} \vec{F}^* \cdot \mathrm{d}\vec{\ell} + \mathrm{i} \int_{\partial A} \vec{F}^* \cdot \mathrm{d}\vec{\ell}_\perp \\
&= \int_A (\vec{\nabla} \times \vec{F}^*)_z\, \mathrm{d}A + \mathrm{i} \int_A \vec{\nabla} \cdot \vec{F}^*\, \mathrm{d}A\,, \qquad \mathrm{d}\vec{\ell}_\perp = \mathrm{d}\vec{\ell} \times \hat{e}_z\,.
\end{aligned} \tag{2}$$

All symbols are defined as in the lecture. You may use your lecture notes, but you must fill in any missing intermediate steps. Show your work!

**(b)** Find the vector field $\vec{F}^*$ corresponding to the function $f(z) = 1/z$ and find the line integrals over $\vec{F}^*$ which correspond to the contour integrals $J_i$ over $f(z)$ with $i = 1$ and $i = 2$ (not $i = 3$).

**Task 3** (50 points). Show that the Green function of the two-dimensional Poisson equation is

$$G(x, y) = \frac{1}{2\pi} \ln\left(\frac{\sqrt{x^2 + y^2}}{a}\right) \qquad \vec{\nabla}^2 G(x, y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) G(x, y) = \delta^{(2)}(x, y) = \delta(x)\, \delta(y)\,. \tag{3}$$

This can be done as follows: First, show that $\nabla^2 G(x, y) = 0$ for any point where $G = G(x, y)$ is differentiable, i.e., for any point excluding the origin. Then, show that the Dirac-$\delta$ term is obtained when one applies the divergence theorem to an infinitesimal circle around the origin.

Furthermore, show that $\vec{\nabla} G(x, y)$ is proportional to $\vec{F}^*(x, y)$, where $\vec{F}^*(x, y)$ is the vector field corresponding to the complex function $1/z$, which was discussed in task 2. Find the proportionality coefficient!

The tasks are due Thursday, 08–FEB–2023.