Task 1 (50 points)
In the lecture, we had defined a rotation matrix $\mathbb{R}(\theta)$ and a projection matrix $\mathbb{P}_x$ as follows:

$$\mathbb{R}(\vartheta) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}, \qquad \mathbb{P}_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1}$$

Calculate the commutator $\mathbb{C} = [\mathbb{R}(\vartheta), \mathbb{P}_x]$ and interpret your result geometrically! In particular, calculate the eigenvalues and eigenvectors of $\mathbb{C}$!

Task 2 (50 points)
In the lecture, we had defined a matrix representation of a complex number z as $\mathbb{M}(z)$. Show that the matrix representation of the reciprocal of a complex number, $\mathbb{M}(z^{-1})$, is equal to the inverse of the matrix representation of z, $[\mathbb{M}(z)]^{-1}$, by calculating both sides of the equation “from first principle”,

$$\mathbb{M}(z^{-1}) = [\mathbb{M}(z)]^{-1} \,. \tag{2}$$

Show your work!

Task 3 (30 points)
Collect all notes you have from previous mathematical preparation courses, on Taylor and Laurent expansions, as well as differential equations and special functions (notably, Bessel functions). Try to refesh your memory on all aspects you may have forgotten in the meantime.

In particular, write a short essay on the question: How would you evaluate a line integral

$$I = \int_P \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{r} \tag{3}$$

where P is a nontrivial path (say, with curves) and $\vec{F}(\vec{r})$ is a vector-valued function of a vector-valued variable $\vec{r}$ (the position). Hint: Start from a parameterisation $\vec{r} = \vec{r}_P(t)$, where t is the time and $\vec{r}_P(t)$ is the position of the object on the path P at time t.

How would your task become easier if

$$\vec{F}(\vec{r}) = -\vec{\nabla} f(\vec{r}) \,, \tag{4}$$

where $f(\vec{r})$ is a scalar function (a potential). (Please note: The minus sign is inserted only for convention and has no special meaning.)

Task 4 (50 points)
Derive the Taylor expansion

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \mathcal{O}(x^4) \,, \tag{5}$$

or at least the first three terms thereof. Explain the meaning of the term $\mathcal{O}(x^4)$. Explain the connection of the Taylor expansion to the Einstein formula

$$E = \sqrt{\vec{p}^2\, c^2 + m^2 c^4} \tag{6}$$

with symbols explained as in the lecture. Where does the nonrelativistic kinetic energy $\vec{p}^2/(2m)$ appear in the formulas? Make a plot, using a computer algebra system of your choice, of the approximations

$$\sqrt{1+x} \approx 1 + \frac{x}{2} \,, \tag{7}$$

$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8} \,, \tag{8}$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} \,, \tag{9}$$

and show, visually, how the approximations become better with increasing order.

The tasks are due Thursday, 01-FEB-2024. Have fun doing the problems!