UNMARKED!

Task 1 (unmarked)
In the lecture, it was shown that in the long-wavelength limit,

$$n_{j\mu} \approx -\frac{\mathrm{i}\, c}{(2j+1)!!} \sqrt{\frac{j+1}{j}}\, k^j\, q_{j\mu}\,. \tag{1}$$

Complete the calculation of the corresponding electric dipole ($j=1$) contribution to $\vec{A}_0(\vec{r})$, based on the representation

$$\vec{A}_0(\vec{r}) = \mathrm{i}k\,\mu_0 \sum_{j=1}^{\infty} \sum_{\mu=-j}^{j} \left(m_{j\mu}\, \vec{M}^{(1)}_{j\mu}(k,\vec{r}) + n_{j\mu}\, \vec{N}^{(1)}_{j\mu}(k,\vec{r}) + l_{j\mu}\, \vec{L}^{(1)}_{j\mu}(k,\vec{r}) \right) \to \sum_{j=1}^{\infty} \sum_{\mu=-j}^{j} n_{j\mu}\, \vec{N}^{(1)}_{j\mu}(k,\vec{r})\,, \tag{2}$$

and verify that you obtain the familiar form of the electric dipole term.

Task 2 (unmarked)
Show that

$$\int \mathrm{d}x\, f_j(x)\, g_{j'}(x) = \frac{x^2}{j'(j'+1) - j(j+1)} \left[f_j(x)\, g'_{j'}(x) - f'_j(x)\, g_{j'}(x) \right]\,, \tag{3}$$

with the help of the differential equation $\left(\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j\,(j+1)}{x^2} - 1 \right) f_j(x) = 0$, where f_j and g_j can be a j_j or y_j Bessel function, or a superposition (spherical Hankel function).

Task 3 (unmarked)
Consider the equation defining the retarded time, $t_{\mathrm{ret}} = t - |\vec{r} - \vec{R}(t_{\mathrm{ret}})|/c$, for uniform motion in only one dimension (along the x direction), i.e., for $\vec{r} = r\,\hat{\mathrm{e}}_x$, $r > 0$, and $\vec{R}(t) = v_0\, t\, \hat{\mathrm{e}}_x$. We thus consider only observation points $\vec{r}$ lying in the x axis, reducing the problem to one dimension.
(a.) Solve the defining equation for t_{ret}, replacing all vector quantities by their x components.
(b.) Calculate the scalar and vector potentials (Liénard–Wiechert potentials),

$$\Phi\,(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\, \frac{1}{|\vec{r} - \vec{R}(t_{\mathrm{ret}})|} \left(1 - \frac{\dot{\vec{R}}(t_{\mathrm{ret}})}{c} \cdot \frac{\vec{r} - \vec{R}(t_{\mathrm{ret}})}{|\vec{r} - \vec{R}(t_{\mathrm{ret}})|} \right)^{-1}\,, \qquad \vec{A}\,(\vec{r},t) = \frac{\Phi\,(\vec{r},t)}{c^2}\, \dot{\vec{R}}(t_{\mathrm{ret}})\,, \tag{4}$$

and show that for the particular case, the final formulas are consistent with “no retardation occurring”, because

$$\Phi\,(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\, \frac{1}{r - v_0\, t}\,, \qquad \vec{A}\,(\vec{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\, \frac{\vec{v}_0}{r - v_0\, t}\,, \tag{5}$$

because the change in the interaction distance $|\vec{r} - \vec{R}(t_{\mathrm{ret}})|$ by the retardation is exactly compensated by the Jacobian factor $\left(1 - \frac{\dot{\vec{R}}(t_{\mathrm{ret}})}{c} \cdot \frac{\vec{r} - \vec{R}(t_{\mathrm{ret}})}{|\vec{r} - \vec{R}(t_{\mathrm{ret}})|} \right)^{-1}$. Show your work and all intermediate steps.

Task 4 (unmarked)
Investigate the scalar and vector potentials (5), but this time for general $\vec{r}$, constant speed $\vec{v}_0$,

$$\vec{R}_0 = \vec{r} - \vec{v}_0\, t\,, \qquad \beta_0 = \frac{\vec{v}_0}{c}\,, \qquad \frac{\vec{R}_0 \cdot \vec{\beta}_0}{|\vec{R}_0|\, \beta_0} = \cos(\theta_0) = 1 - \frac{1}{2!}\,\theta_0^2 + \frac{1}{4!}\,\theta_0^2 + \mathcal{O}(\theta_0^6)\,. \tag{6}$$

Show that the first retardation correction for small θ_0 reads as follows,

$$\Phi\,(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\, \frac{1}{|\vec{r} - \vec{v}_0\, t|} \left(1 + \frac{\beta_0^2\, \theta_0^2}{2} + \mathcal{O}(\theta_0^4) \right)\,. \tag{7}$$

Hint: The formulas $r - v_0\, t_{\rm ret} = c\,(t - t_{\rm ret})$, with $t_{\rm ret} = (c\,t - r)/(c - v_0)$, may be helpful. For the courageous, unmarked: Calculate the term of order θ_0^4. You may use lecture notes.

Task 5 (unmarked). Start from the dispersion relation for electromagnetic waves in a waveguide,

$$k = \frac{\sqrt{\epsilon_r \mu_r}}{c} \sqrt{\omega^2 - \omega_{mn}^2}\,, \qquad \omega_{mn} = \frac{\pi c}{\sqrt{\epsilon_r \mu_r}} \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 \right]^{1/2} . \tag{8}$$

For each mode, the k_{mn} varies with frequency $\omega > \omega_{mn}$. The ω_{mn} is the cutoff frequency for the mode. Question: How can you generate "slow light" (as far as the group velocity is concerned) in a waveguide? For which frequencies ω does the group velocity become zero?

Task 6 (unmarked) (a) We consider a rectangular resonator. With spherical coordinates θ and φ as defined in the lecture, we write the vector $\vec{k}$ as follows,

$$k_x = k\,\sin\theta\,\cos\varphi\,, \qquad k_y = k\,\sin\theta\,\sin\varphi\,, \qquad k_z = k\,\cos\theta\,, \qquad \vec{k} = k_x\,\hat{e}_x + k_y\,\hat{e}_y + k_z\,\hat{e}_z\,. \tag{9}$$

(i) Find k, θ and φ as a function of k_x, k_y, and k_z! (ii) Write the unit vector $\hat{k} = \vec{k}/|\vec{k}|$ as a function of the spherical coordinates $k = |\vec{k}|$, θ and φ. Hint: This task is easy and basic but still needs to be done!
(b) Define the two polarization vectors

$$\hat{e}_{\vec{k},\lambda=\mathrm{TE}} = \sin\varphi\,\hat{e}_x - \cos\varphi\,\hat{e}_y\,, \qquad \hat{e}_{\vec{k},\lambda=\mathrm{TM}} = -\cos\theta\,\cos\varphi\,\hat{e}_x - \cos\theta\sin\varphi\,\hat{e}_y + \sin\theta\,\hat{e}_z\,, \tag{10}$$

where the θ and φ are the same as in equation (9). Show the relations

$$\vec{k}\cdot\hat{e}_{\vec{k},\lambda=\mathrm{TE}} = 0 \qquad \vec{k}\cdot\hat{e}_{\vec{k},\lambda=\mathrm{TM}} = 0 \qquad \hat{e}_{\vec{k},\lambda=\mathrm{TM}} \times \hat{e}_{\vec{k},\lambda=\mathrm{TE}} = \hat{k} = \vec{k}/|\vec{k}|\,. \tag{11}$$

(c) Define the TE and TM vector potentials ($\mathcal{A}_0$ is a global amplitude)

$$\left(A_{\vec{k},\lambda}\right)_x = \mathcal{A}_0\,\sqrt{8/V}\,\left(\hat{e}_{\vec{k},\lambda}\right)_x\,\cos(k_x x)\,\sin(k_y y)\,\sin(k_z z)\,\exp(-\mathrm{i}\,\omega\,t)\,, \tag{12a}$$

$$(A_{\vec{k}\lambda})_y = \mathcal{A}_0\,\sqrt{8/V}\,\left(\hat{e}_{\vec{k},\lambda}\right)_y\,\sin(k_x x)\,\cos(k_y y)\,\sin(k_z z)\,\exp(-\mathrm{i}\,\omega\,t)\,, \tag{12b}$$

$$(A_{\vec{k}\lambda})_z = \mathcal{A}_0\,\sqrt{8/V}\,\left(\hat{e}_{\vec{k},\lambda}\right)_z\,\sin(k_x x)\,\sin(k_y y)\,\cos(k_z z)\,\exp(-\mathrm{i}\,\omega\,t)\,. \tag{12c}$$

Here, the Cartesian components are denoted by subscripts x, y and z, and $\lambda = \mathrm{TM}$ or $\lambda = \mathrm{TE}$.

(i) Calculate $\vec{E} = -\frac{\partial}{\partial t}\vec{A}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$ by explicit differentiation using Cartesian coordinates. Hint: Keep k_x, k_y and k_z in symbolic form. The angles θ and φ belong to $\vec{k}$, not $\vec{r}$, and you do not need to differentiate them! **(ii)** Show that $\vec{\nabla}\cdot\vec{A} = 0$ and separately that $k_x^2 + k_y^2 + k_z^2 = \epsilon_r\,\mu_r \omega^2/c^2$. **(iii)** Show that the electric field corresponding to the polarization $\lambda = \mathrm{TE}$ is transverse, i.e., its z component vanishes. **(iv)** Show that the magnetic induction field corresponding to the polarization $\lambda = \mathrm{TM}$ is transverse, i.e., its z component vanishes. Hint: Consider both equations (9) and (12). **(v)** Show that the boundary conditions for the electric field at the boundaries of the rectangular cavity are fulfilled provided you choose (explain your reasoning in full English sentences for full credit!)

$$k_x = \frac{m\pi}{L_x}\,, \qquad k_y = \frac{n\pi}{L_y}\,, \qquad k_z = \frac{p\pi}{L_z}\,, \qquad m,n,p \in \mathbb{N}_0 \tag{13}$$

where the edge lengths of the rectangular cavity are L_z, L_y, and L_z (in the x, y, and z directions). **(vi)** For $L_x < L_y < L_z$, determine the quantum numbers m, n and p for the fundamental angular frequency of the cavity. Hint: Consider what happens when two quantum numbers vanish. Then, perhaps, choose one quantum number to vanish which would otherwise give the shortest wavelength=highest frequency. This might be the one that corresponds to the shortest edge length, i.e., a.

Task 7 (unmarked)
We investigate a simple model of a frequency-dependent dielectric constant. To this end we write

$$\widetilde{\epsilon}(\omega) = \epsilon_0 \left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - \mathrm{i}\,\omega\,\gamma} \right)\,. \tag{14}$$

(a) Verify that $\widetilde{\epsilon}(-\omega) = \widetilde{\epsilon}(\omega)^*$.

(b) Show that the relation $\widetilde{\vec{D}}(\omega) = \widetilde{\epsilon}(\omega)\,\widetilde{\vec{E}}(\omega)$ implies $\vec{D}(t) = \int_{-\infty}^{\infty} \mathrm{d}t'\, \epsilon(t-t')\, \vec{E}(t')$. where

$$\epsilon(\tau) = \epsilon_0\, \delta(\tau) + \epsilon_0\, G(\tau)\,, \qquad G(\tau) = \int \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\,\omega\,\tau}\, \frac{\omega_p^2}{\omega_0^2 - \omega^2 - \mathrm{i}\,\omega\,\gamma} \tag{15}$$

(c) Show, with reference to the location of the poles in the complex plane, that

$$G(\tau) = \exp\left(-\tfrac{1}{2}\,\gamma\,\tau\right)\, \omega_p^2\, \frac{\sin(\tau\,\nu_0)}{\nu_0}\, \Theta(\tau)\,, \qquad \nu_0 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}\,. \tag{16}$$

(d) Show that

$$\vec{D}(t) = \epsilon_0\, \vec{E}(\vec{r}, t) + \epsilon_0 \int_0^\infty \mathrm{d}\tau\, G(\tau)\, \vec{E}(t-\tau) \tag{17}$$

and interpret your result in terms of causality.

Task 8 (unmarked)
Show that for a dielectric constant of a dilute gas, the Clausius–Mosotti equation $\frac{\tilde{\epsilon}_r(\omega)-1}{\tilde{\epsilon}_r(\omega)+2} = \frac{N_V\,\alpha(\omega)}{3\,\epsilon_0}$ and the previously derived result for a dilute gas $\tilde{\epsilon}_r(\omega) = 1 + \frac{N_V}{\epsilon_0}\,\alpha(\omega)$ are in agreement. Hint: Set $\tilde{\epsilon}_r(\omega) = 1 + \chi_e(\omega)$ and perform a Taylor expansion in $\chi_e(\omega)$. For the denominator, you may be able to approximate $\tilde{\epsilon}_r(\omega) + 2 \approx 3$. Hint: The linear term is decisive for this exercise. For full credit, explain your reasoning!

Task 9 (unmarked)
How would you have to modify the Clausius–Mosotti equation for a mixture of molecules with molar concentration $N_V^{(1)}$, $N_V^{(2)}$, and $N_V^{(3)}$, and polarizabilities $\alpha^{(1)}$, $\alpha^{(2)}$, and $\alpha^{(3)}$? Hint: The answer is quite obvious but still worthy to be written down. For full credit, explain your reasoning!

Task 10 (unmarked)
Based on the Clausius-Mosotti equation, $\frac{\tilde{\epsilon}_r(\omega)-1}{\tilde{\epsilon}_r(\omega)+2} = \frac{N_V\,\alpha(\omega)}{3\,\epsilon_0}$, show [e.g., by taking appropriate logarithms] that

$$\frac{\mathrm{d}\tilde{\epsilon}_r}{\mathrm{d}N_V} = \frac{(\tilde{\epsilon}_r - 1)\,(\tilde{\epsilon}_r + 2)}{3\,N_V}\,. \tag{18}$$

Task 11 (unmarked)
Find the saddle-point approximation to the integral

$$I = \int_{-\infty}^{\infty} \mathrm{d}x\, \exp\left(-4 - 8\,x^2 - x^4\right) \tag{19}$$

and compare the result with the exact numerical value. Hint: Do not miss the *three* saddle points, one might be at the origin, and the location of the other two would need to be determined. Choose the "lowest mountain pass" and determine the direction in the complex plane in which you approach the saddle point.

Task 12 (unmarked)
For a plasma, assume that

$$\tilde{\epsilon}_r(\omega) = 1 - \frac{\sigma_0}{\epsilon_0}\, \frac{1}{\omega\;(\omega\,\tau_0 + \mathrm{i})} \tag{20}$$

and calculate the Fourier (back-)transform

$$\epsilon_r(\tau) = \delta(\tau) + \Theta(\tau)\, \frac{\sigma_0}{\epsilon_0} \left[1 - \exp\left(-\frac{\tau}{\tau_0}\right)\right]. \tag{21}$$

Hint: The pole at $\omega = 0$ requires special treatment. Due to the requirement of causality, you have to write Eq. (20) as

$$\tilde{\epsilon}_r(\omega) = 1 - \frac{\sigma_0}{\epsilon_0}\, \frac{1}{(\omega + \mathrm{i}\,\eta)\;(\omega\,\tau_0 + \mathrm{i})}\,, \qquad \eta \to 0^+\,, \tag{22}$$

where $\eta \to 0^+$ is a small, positive, infinitesimal parameter.

Task 13 (unmarked)
In general, the Kramers-Kronig relations allow us to calculate the real part of $\tilde{\epsilon}_r(\omega)$ given only the imaginary part of $\tilde{\epsilon}_r(\omega)$ for positive ω. Consider a model with

$$\mathrm{Im}\, \tilde{\epsilon}_r(\omega) = \lambda\, [\Theta(\omega - \omega_1) - \Theta(\omega - \omega_2)]\,, \qquad \omega_2 > \omega_1 > 0\,. \qquad \omega > 0\,. \tag{23}$$

(For $\omega < 0$, we have $\mathrm{Im}\, \tilde{\epsilon}_r(\omega) = -\mathrm{Im}\, \tilde{\epsilon}_r(-\omega)$, commensurate with the causal behavior.) This function is equal to λ for $\omega_1 < \omega < \omega_2$ and zero otherwise. Which physical dimension must λ have? Using the Kramers–Kronig relations, calculate $\mathrm{Re}[\tilde{\epsilon}_r(\omega)]$. Sketch the behavior of $\mathrm{Im}[\tilde{\epsilon}_r(\omega)]$ and $\mathrm{Re}[\tilde{\epsilon}_r(\omega)]$ as a function of ω.

Task 14 (unmarked)
We recall the simple model of a frequency-dependent dielectric constant. To this end we write

$$\tilde{\epsilon}(\omega) = \epsilon_0 \left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - \mathrm{i}\,\omega\,\gamma}\right)\,. \tag{24}$$

Show that $\epsilon(\omega)$ fulfills the following Kramers–Kronig relations,

$$\mathrm{Re}(\tilde{\epsilon}_r(\omega)) = 1 + \frac{1}{\pi}\,(\mathrm{P.V.}) \int_{-\infty}^{\infty} \frac{\mathrm{Im}(\tilde{\epsilon}_r(\omega'))}{\omega' - \omega}\, \mathrm{d}\omega'\,, \tag{25a}$$

$$\mathrm{Im}(\tilde{\epsilon}_r(\omega)) = -\frac{1}{\pi}\,(\mathrm{P.V.}) \int_{-\infty}^{\infty} \frac{\mathrm{Re}(\tilde{\epsilon}_r(\omega')) - 1}{\omega' - \omega}\, \mathrm{d}\omega'\,. \tag{25b}$$

Your explicit calculation may proceed by identifying a complex integration contour that replaces the integral from $-\infty$ to $+\infty$, with all relevant poles and corresponding residues taken into account.

Task 15 (unmarked)
Re-familiarize yourself with the definition of a principal-value integral. Convince yourself that

$$(\mathrm{P.V.}) \int_{-a}^{a} \mathrm{d}t\, \frac{1}{t-b} = \lim_{\delta \to 0^+} \int\limits_{-a}^{b-\delta} \mathrm{d}t\, \frac{1}{t-b} + \int\limits_{b+\delta}^{a} \mathrm{d}t\, \frac{1}{t-b} = \ln\left(\left|\frac{a-b}{a+b}\right|\right)\,. \tag{26}$$

Calculate

$$(\mathrm{P.V.}) \int_{-1}^{1} \mathrm{d}t\, \frac{1}{t-0.3} \approx \int\limits_{-1}^{0.3-10^{-6}} \mathrm{d}t\, \frac{1}{t-0.3} + \int\limits_{0.3+10^{-6}}^{1} \mathrm{d}t\, \frac{1}{t-0.3} \tag{27}$$

by numerical integration on a computer. Then, compare the two results obtained, numerically, for $a = 1$ and $b = 0.3$. You may use internet interfaces to computer algebra software but you must include a protocol (printout) of the computer session.

Task 16 (unmarked)
Consider the formula [with $\tilde{\epsilon}_r^{(\infty)} = 1$]

$$\tilde{\epsilon}_r(\omega) = \tilde{\epsilon}_r^{(\infty)} + \omega_{\mathrm{T}}^2\, \frac{\tilde{\epsilon}_r^{(0)} - \tilde{\epsilon}_r^{(\infty)}}{\omega_{\mathrm{T}}^2 - \omega^2 - \mathrm{i}\,\gamma\,\omega} = 1 + \omega_{\mathrm{T}}^2\, \frac{\tilde{\epsilon}_r^{(0)} - 1}{\omega_{\mathrm{T}}^2 - \omega^2 - \mathrm{i}\,\gamma\,\omega} \tag{28}$$

for the frequency-dependent dielectric constant of a solid, in the limit $\gamma \to 0$. Your task is to "match" the parameters α and ω_0 to the familiar representations for the real and imaginary parts of the dielectric constant,

$$\mathrm{Re}\, \tilde{\epsilon}_r(\omega) = 1 + \frac{\alpha\,\omega_0^2}{\omega_0^2 - \omega^2}\,, \qquad \mathrm{Im}\, \tilde{\epsilon}_r(\omega) = \frac{\pi}{2}\,\alpha\,\omega_0\,\delta(\omega - \omega_0) - \frac{\pi}{2}\,\alpha\,\omega_0\,\delta(\omega + \omega_0)\,. \tag{29}$$

I.e., write the parameters α and ω_0 as a function of $\tilde{\epsilon}_r^{(0)}$ and ω_{T}.
Hint: You may want to apply the Sokhotskii–Plemelj prescription

$$\frac{1}{x - \mathrm{i}\eta} = (\mathrm{P.V.})\,\frac{1}{x} + \mathrm{i}\,\pi\,\delta(x)\,, \qquad \eta \to 0^+\,, \tag{30}$$

which is valid for infinitesimal positive $\eta \to 0^+$.

Task 17 (unmarked)
Under a Lorentz boost into a frame that moves with velocity $\vec{v}$ (vector), we have

$$ct' = \gamma\,ct - \left(\gamma\,\frac{v}{c}\right)(\hat{v}\cdot\vec{r})\,, \qquad \vec{r}' = \vec{r} + (\gamma - 1)\,\frac{\vec{v}\cdot\vec{r}}{\vec{v}^{\,2}}\,\vec{v} - \left(\gamma\,\frac{v}{c}\right)\hat{v}\,(ct)\,. \tag{31}$$

Show that $(ct')^2 - \vec{r}'^2 = (ct)^2 - \vec{r}^2$ under the Lorentz boost. Here, $\hat{v} = \vec{v}/|\vec{v}|$ is the unit vector in the $\vec{v}$ direction. Energy and momentum transform as follows,

$$\begin{aligned} E' &= \gamma\,E - \left(\gamma\,\frac{v}{c}\right)\,\hat{v}\cdot(c\vec{p}) = \gamma\,\left(E - \vec{v}\cdot\vec{p}\right)\,, \\ c\vec{p}' &= c\vec{p} + (\gamma - 1)\,\frac{\vec{v}\cdot(c\vec{p})}{\vec{v}^2}\,\vec{v} - \left(\gamma\,\frac{v}{c}\right)\hat{v}\,E. \end{aligned} \tag{32}$$

Show that $(E')^2 - (c\vec{p}')^2 = E^2 - (c\vec{p})^2$.

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