This exercise sheet summarizes important formulas and can be used as a reference sheet as well. We know that the coupling of the vector potential to the current density reads as

$$\vec{A}_0(\vec{r}) = \frac{1}{c^2} \int \mathrm{d}^3 r' \, \mathbb{G}_R(k, \vec{r} - \vec{r}') \cdot \vec{J}_0(\vec{r}') \,. \tag{1}$$

We had encountered three ways to expand the potential into multipole moments.

(i). First Expansion. The first expansion reads as follows:

$$\mathbb{G}_R(\vec{r} - \vec{r}', k) = \frac{1}{\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathrm{i}\, k \, j_\ell \left(k \, r_<\right) \, h_\ell^{(1)} \left(k \, r_>\right) \, Y_{\ell m} \left(\theta, \varphi\right) \, Y_{\ell m}^* \left(\theta', \varphi'\right) \, \mathbb{1}_{3\times 3} \,. \tag{2}$$

We define multipole moments and obtain the vector potential is obtained as follows,

$$\vec{p}_{\ell m} = \int \mathrm{d}^3 r \, j_\ell(k\, r) \, \vec{J}_0(\vec{r}) \, Y_{\ell m}^*(\theta, \varphi) \,, \qquad \vec{A}_0(\vec{r}) = \mathrm{i}\, \mu_0 \, k \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \vec{p}_{\ell m} \, h_\ell^{(1)}(k\, r) \, Y_{\ell m}(\theta, \varphi) \,. \tag{3}$$

(ii). Second Expansion. The second expansion involves the vector spherical harmonics,

$$\mathbb{G}_R(\vec{r} - \vec{r}', k) = \frac{\mathrm{i}\, k}{\epsilon_0} \sum_{j=0}^{\infty} \sum_{\mu=-j}^{j} \sum_{\ell=j-1}^{j+1} j_\ell(k\, r_<) \, h_\ell^{(1)}(k\, r_>) \, \vec{Y}_{j\mu}^{\ell}(\theta, \varphi) \otimes \vec{Y}_{j\mu}^{\ell *}(\theta', \varphi') \,. \tag{4}$$

The multipole moments read as follows,

$$p_{j\mu}^{\ell} = \int \mathrm{d}^3 r \, j_\ell(k\, r) \, \vec{J}_0(\vec{r}) \cdot \vec{Y}_{j\mu}^{\ell *}(\theta, \varphi) \,, \qquad \vec{A}_0(\vec{r}) = \mathrm{i}\, \mu_0 \, k \sum_{j=0}^{\infty} \sum_{\mu=-j}^{j} \sum_{\ell=j-1}^{j+1} p_{j\mu}^{\ell} \, h_\ell^{(1)}(k\, r) \, \vec{Y}_{j\mu}^{\ell}(\theta, \varphi) \,. \tag{5}$$

(iii). Third Expansion. The third expansion involves the Helmholtz Green function,

$$\begin{aligned} \mathbb{G}_R(k, \vec{r} - \vec{r}') = \frac{\mathrm{i}\, k}{\epsilon_0} \sum_{j=0}^{\infty} \sum_{\mu=-j}^{j} \Big(& \vec{M}_{j\mu}^{(1)}(k, \vec{r}_>) \otimes \vec{M}_{j\mu}^{(0)*}(k, \vec{r}_<) \\ & + \vec{N}_{j\mu}^{(1)}(k, \vec{r}_>) \otimes \vec{N}_{j\mu}^{(0)*}(k, \vec{r}_<) + \vec{L}_{j\mu}^{(1)}(k, \vec{r}_>) \otimes \vec{L}_{j\mu}^{(0)*}(k, \vec{r}_<) \Big) \,. \end{aligned} \tag{6}$$

The magnetic and electric multipole, and logitudinal, moments are

$$m_{j\mu} = \int \mathrm{d}^3 r' \, \vec{J}_0(\vec{r}') \cdot \vec{M}_{j\mu}^{(0)*}(\theta', \varphi') \,, \quad n_{j\mu} = \int \mathrm{d}^3 r' \, \vec{J}_0(\vec{r}') \cdot \vec{N}_{j\mu}^{(0)*}(\theta', \varphi'), \quad l_{j\mu} = \int \mathrm{d}^3 r' \, \vec{J}_0(\vec{r}') \cdot \vec{L}_{j\mu}^{(0)*}(\theta', \varphi'). \tag{7}$$

The vector potential is obtained as

$$\vec{A}_0(\vec{r}) = \mathrm{i} k \, \mu_0 \sum_{j=0}^{\infty} \sum_{\mu=-j}^{j} \left(m_{j\mu} \, \vec{M}_{j\mu}^{(1)}(k, \vec{r}) + n_{j\mu} \, \vec{N}_{j\mu}^{(1)}(k, \vec{r}) + l_{j\mu} \, \vec{L}_{j\mu}^{(1)}(k, \vec{r}) \right) \,. \tag{8}$$

The magnetic, electric and longitudinal multipole functions are as follows (please turn the page over!),

$$\vec{M}_{j\mu}^{(K)}(k, \vec{r}) = \frac{1}{\sqrt{j(j+1)}} \, f_j^{(K)}(k\, r) \, \vec{L} Y_{j\mu}(\theta, \varphi) \,, \tag{9a}$$

$$\vec{N}_{j\mu}^{(K)}(k, \vec{r}) = \frac{\mathrm{i}}{k} \, \vec{\nabla} \times \vec{M}_{j\mu}^{(K)}(k, \vec{r}) \,, \tag{9b}$$

$$\vec{L}_{j\mu}^{(K)}(k, \vec{r}) = \frac{1}{k} \, \vec{\nabla} \left(f_j^{(K)}(k\, r) \, Y_{j\mu}(\theta, \varphi) \right) \,. \tag{9c}$$

Task 1 (100 points)
Consider all three variants of the multipole expansion, given in Eqs. (2), (4) and (6), under the following conditions: First, set

$$r_< = r' , \qquad \vec{r}' = \vec{0} , \qquad r' = 0 . \tag{10}$$

That is, you set $r' = 0$ explicitly. Which (quite drastic) simplifications result for Eqs. (2), (4) and (6), under this assumption? Hint: Maybe, the sum over ℓ and m (j and μ) might collapse to a single term, under this assumption? Then, convince yourself that Eqs. (2), (4) and (6), are fulfilled in the case $r' = 0$. To this end, you might have to look up the formula for $\mathbb{G}_R(k, \vec{r} - \vec{r}')$ in the lecture notes, for the left-hand side. I.e., show that, for $r' = 0$, the left-hand side (LHS) and right-hand side (RHS) of Eqs. (2), (4) and (6) fulfill

$$\text{LHS} = \text{RHS} = \frac{\exp(\mathrm{i}kr)}{4\pi\epsilon_0 r} \mathbb{1}_{3\times 3} . \qquad r' = 0 . \tag{11}$$

Task 2 (100 points)
Please treat the following current distributions in tasks 1, 2, and 3, using the $\vec{M}^{(K)}$, $\vec{N}^{(K)}$, and $\vec{L}^{(K)}$ functions. You may use computer algebra if convenient. In all derivations, you can replace the spherical Bessel functions (but not the Hankel functions) by their short-range asymptotics.
Assume that

$$\vec{J}_0(\vec{r}) = \frac{I_0}{a^2} \left(3\frac{z^2}{r^2} - 1 \right) \exp\left(-\frac{r^2}{a^2} \right) \hat{\mathrm{e}}_z , \tag{12}$$

and use all three multipole formalisms. You may replace the Bessel functions by their leading asymptotic behavior for small argument.
Hint: You might obtain

$$n_{10} = -\frac{\pi}{25\sqrt{6}} k^2\, a^3\, I_0 , \qquad n_{30} = -\frac{\pi}{25} \sqrt{\frac{3}{7}}\, k^2\, a^3\, I_0 , \tag{13}$$

$$l_{10} = -\frac{\pi}{25\sqrt{3}} k^2\, a^3\, I_0 , \qquad l_{30} = \frac{3\pi}{50\sqrt{7}} k^2\, a^3\, I_0 . \tag{14}$$

Do all the magnetic multipoles vanish, i.e., $m_{j\mu} = 0$?
Task 3 (100 points)
Assume that

$$\vec{J}_0(\vec{r}) = \frac{I_0}{a^2} \left(3\frac{z^2}{r^2} - 1 \right) \exp\left(-\frac{r^2}{a^2} \right) \vec{e}_{+1} . \tag{15}$$

Hint: You might obtain

$$m_{21} = -\frac{\pi}{10\sqrt{10}} k^2\, a^3\, I_0 , \qquad n_{11} = \frac{\pi}{50\sqrt{6}} k^2\, a^3\, I_0 , \tag{16}$$

$$n_{31} = -\frac{\pi}{25} \sqrt{\frac{2}{7}}\, k^2\, a^3\, I_0 , \qquad l_{11} = \frac{\pi}{50\sqrt{3}} k^2\, a^3\, I_0 , \tag{17}$$

$$l_{31} = \frac{\pi}{25} \sqrt{\frac{3}{14}}\, k^2\, a^3\, I_0 . \tag{18}$$

Do all the other multipoles vanish, i.e., $m_{j\mu} = 0$?
Task 4 (100 points)
Now here is another current distribution,

$$\vec{J}_0(\vec{r}) = \frac{I_0}{a^2} \frac{z}{a} \left(\frac{y}{a} \hat{\mathrm{e}}_x - \frac{x}{a} \hat{\mathrm{e}}_y \right) \exp\left(-\frac{r^2}{a^2} \right) . \tag{19}$$

Hint: You might obtain

$$m_{20} = \mathrm{i} \frac{\pi}{4\sqrt{30}} k^2\, a^3\, I_0 . \tag{20}$$

Do all the electric and longitudinal multipoles vanish?

The tasks are due Thursday, 07–DEC–2023,
with a possible extension to Saturday, 09–DEC–2023.