Task 1 (30 points). Let $\vec{v} = v_x\,\hat{e}_x + v_y\,\hat{e}_y + v_z\,\hat{e}_z$. Calculate, with definitions as in the lecture,

$$\mathbb{T}|_{\ell=0} = (\vec{v} \otimes \vec{v})_{\ell=0}\,, \qquad \mathbb{T}|_{\ell=1} = (\vec{v} \otimes \vec{v})_{\ell=1}\,, \qquad \mathbb{T}|_{\ell=2} = (\vec{v} \otimes \vec{v})_{\ell=2}\,, \tag{1}$$

in terms of the components v_x, v_y and v_z.

Write a short essay on the connection between *(i)* the $(\ell = 1)$-component of the tensor product of two vectors, *(ii)* the vector cross product, and *(iii)* Clebsch–Gordon coefficients. Use the lecture notes for this task.

Task 2 (30 points). We had discussed the spin-1 spin matrices in the lecture. Show by explicit calculations, using the explicit form of $\$_3$, that

$$\$_3\,\vec{e}_q = q\,\vec{e}_q\,, \qquad q = -1, 0, 1\,. \tag{2}$$

Task 3 (30 points). In the lecture, with a notation clarified therein, we had shown the following relations for the scalar and vector potentials generated by an oscillating source,

$$\Phi_0(\vec{r}) = \frac{\mathrm{i}k}{\epsilon_0} \sum_{j=0}^{\infty} \sum_{\mu=-j}^{j} p_{j\mu}\, h_j^{(1)}(k\,r)\, Y_{j\mu}(\theta, \varphi)\,, \tag{3}$$

$$\vec{A}_0(\vec{r}) = \mathrm{i}k\,\mu_0 \sum_{j=0}^{\infty} \sum_{\mu=-j}^{j} \left(m_{j\mu}\, \vec{M}_{j\mu}^{(1)}(k, \vec{r}) + n_{j\mu}\, \vec{N}_{j\mu}^{(1)}(k, \vec{r}) + l_{j\mu}\, \vec{L}_{j\mu}^{(1)}(k, \vec{r}) \right)\,. \tag{4}$$

Show that the Lorentz gauge condition is fulfilled, which reads in the mixed frequency-coordinate representation,

$$\vec{\nabla} \cdot \vec{A}_0(\vec{r}) - \frac{\mathrm{i}\omega}{c^2} \Phi_0(\vec{r}) = 0\,. \tag{5}$$

Task 4 (30 points). For an oscillating source, the vector potential reads as follows,

$$\vec{A}_0(\vec{r}) = \mathrm{i}k\,\mu_0 \sum_{j=0}^{\infty} \sum_{\mu=-j}^{j} \left(m_{j\mu}\, \vec{M}_{j\mu}^{(1)}(k, \vec{r}) + n_{j\mu}\, \vec{N}_{j\mu}^{(1)}(k, \vec{r}) + l_{j\mu}\, \vec{L}_{j\mu}^{(1)}(k, \vec{r}) \right)\,. \tag{6}$$

The electric field could be obtained in two ways. In the first way, one simply writes

$$\text{(Way 1:)} \qquad \vec{A}_{0\perp}(\vec{r}) = \mathrm{i}k\,\mu_0 \sum_{j=0}^{\infty} \sum_{\mu=-j}^{j} \left(m_{j\mu}\, \vec{M}_{j\mu}^{(1)}(k, \vec{r}) + n_{j\mu}\, \vec{N}_{j\mu}^{(1)}(k, \vec{r}) \right)\,, \qquad \vec{E}_0(\vec{r}) = \mathrm{i}\omega \vec{A}_{0\perp}(\vec{r})\,. \tag{7}$$

In the second way, one calculates

$$\text{(Way 2:)} \qquad \vec{B}_0(\vec{r}) = \vec{\nabla} \times \vec{A}_0(\vec{r})\,, \qquad \vec{E}_0'(\vec{r}) = \frac{\mathrm{i}c^2}{\omega}\, \vec{\nabla} \times \vec{B}_0(\vec{r})\,. \tag{8}$$

Show the $\vec{E}_0(\vec{r}) = \vec{E}_0'(\vec{r})$.

The tasks are due Tuesday, 28–NOV–2023. Extensions, if at all, will be very limited.