Task 1 (30 points). In the lecture, we have encountered the Clebsch–Gordan coefficients. Show that

$$s = \sum_{q' q''} C^{00}_{1q'\,1q''}\, u_{q'}\, v_{q''} = -\frac{1}{\sqrt{3}}\,\vec{u}\cdot\vec{v}\,, \tag{1}$$

where the components u_q and v_q are given in the spherical basis, i.e., we would have for the reference vector $\vec{r} = \sum x_q\,\vec{\mathrm{e}}^{\,*}_q$,

$$x_{+1} = -\frac{1}{\sqrt{2}}\,(x + \mathrm{i}\,y)\,, \qquad x_0 = z\,, \qquad x_{-1} = \frac{1}{\sqrt{2}}\,(x - \mathrm{i}\,y)\,, \tag{2}$$

with

$$\vec{\mathrm{e}}_{+1} = -\frac{1}{\sqrt{2}}\,(\hat{e}_x + \mathrm{i}\,\hat{e}_y)\,, \qquad \vec{\mathrm{e}}_0 = \hat{e}_z\,, \qquad \vec{\mathrm{e}}_{-1} = \frac{1}{\sqrt{2}}\,(\hat{e}_x - \mathrm{i}\,\hat{e}_y)\,. \tag{3}$$

Which summation limits have to be used for the sums over q' and q''?

Task 2 (30 points). Consider the relations,

$$\vec{Y}^{j-1}_{j\,\mu}(\theta,\varphi) = -\frac{1}{\sqrt{j\,(2j+1)}}\left(\mathrm{i}\,\hat{r}\times\vec{L} - j\,\hat{r}\right) Y_{j\mu}(\theta,\varphi)\,, \tag{4a}$$

$$\vec{Y}^{j+1}_{j\,\mu}(\theta,\varphi) = -\frac{1}{\sqrt{(j+1)\,(2j+1)}}\left(\mathrm{i}\,\hat{r}\times\vec{L} + (j+1)\,\hat{r}\right) Y_{j\mu}(\theta,\varphi)\,. \tag{4b}$$

With the help of these relations, and the recursion relations for the Bessel functions, show that

$$\begin{aligned}\vec{N}^{(K)}_{j\mu}(k,\vec{r}) &= -\sqrt{\frac{j+1}{2j+1}}\,f^{(K)}_{j-1}(k\,r)\,\vec{Y}^{j-1}_{j\,\mu}(\theta,\varphi) + \sqrt{\frac{j}{2j+1}}\,f^{(K)}_{j+1}(k\,r)\,\vec{Y}^{j+1}_{j\,\mu}(\theta,\varphi)\,,\\ &= \frac{1}{kr}\left\{\frac{\mathrm{d}[kr\,f^{(K)}_j(k\,r)]}{\mathrm{d}(kr)}[\mathrm{i}\hat{r}\times\vec{Y}^j_{j\mu}(\theta,\varphi)] - \hat{r}\sqrt{j(j+1)}f^{(K)}_j(k\,r)\,Y_{j\mu}(\theta,\varphi)\right\},\end{aligned} \tag{5}$$

and that

$$\begin{aligned}\vec{L}^{(K)}_{j\mu}(k,\vec{r}) &= \sqrt{\frac{j}{2j+1}}\,f^{(K)}_{j-1}(k\,r)\,\vec{Y}^{j-1}_{j\,\mu}(\theta,\varphi) + \sqrt{\frac{j+1}{2j+1}}\,f^{(K)}_{j+1}(k\,r)\,\vec{Y}^{j+1}_{j\,\mu}(\theta,\varphi)\\ &= \hat{r}\,\frac{\mathrm{d}[f^{(K)}_j(k\,r)]}{\mathrm{d}(k\,r)}\,Y_{j\mu}(\theta,\varphi) - \sqrt{j(j+1)}\,(\mathrm{i}\hat{r}\times\vec{Y}^j_{j\mu}(\theta,\varphi))\,\frac{1}{k\,r}\,f^{(K)}_j(k\,r)\,.\end{aligned} \tag{6}$$

Task 4 (30 points). Show that

$$\sqrt{\frac{j+1}{2j+1}}\,\vec{Y}^{j-1}_{j\,\mu}(\theta,\varphi) + \sqrt{\frac{j}{2j+1}}\,\vec{Y}^{j+1}_{j\,\mu}(\theta,\varphi) = -\mathrm{i}\left(\hat{r}\times\vec{Y}^j_{j\,\mu}(\theta,\varphi)\right). \tag{7}$$

Task 5 (30 points). Show that

$$\vec{\nabla}\times\left(\vec{r}\times\vec{\nabla}\right) f(\vec{r}) = \left[\vec{r}\,\vec{\nabla}^2 - \vec{\nabla}\,\frac{\partial}{\partial r}\,r\right] f(\vec{r})\,, \tag{8}$$

for *any* test function $f(\vec{r})$. (The test function $f(\vec{r})$ is not necessarily radially symmetric.)

The tasks are due Tuesday, 28–NOV–2023. Extensions, if at all, will be very limited.