Task 1 (50 points)
Consider the angular momentum expansion

$$\frac{\exp\left(\mathrm{i}\,k\,|\vec{r}-\vec{r}'|\right)}{4\pi\,|\vec{r}-\vec{r}'|} = \lim_{\ell_{\max}\to\infty} \sum_{\ell=0}^{\ell_{\max}} \sum_{m=-\ell}^{\ell} \mathrm{i}\,k\,j_\ell\left(k\,r_<\right)\,h^{(1)}_\ell\left(k\,r_>\right)\,Y_{\ell m}\left(\theta,\varphi\right)\,Y^*_{\ell m}\left(\theta',\varphi'\right)\,, \tag{1}$$

for $\vec{r} = 2\hat{e}_x + 3\hat{e}_y + 5\hat{e}_z$, and $\vec{r}' = 12\,\hat{e}_z$, and $k = 5/3$ [we consider Eq. (1) as a mathematical identity and neglect physical units]. Calculate θ, φ, θ' and φ'. Then, convince yourself that only the $m = 0$ terms in Eq. (1) contribute (why?) and use your favorite computer algebra system to calculate the first 50 partial sums ($\ell_{\max} = 1, \dots, 50$) of the series on the right-hand side of Eq. (1), and observe to which accuracy the series on the right-hand side of Eq. (1) approximates the exact value on the left-hand side of Eq. (1).

Task 2 (20 points)
Show, by reference to the differential equation fulfilled by j_ℓ and y_ℓ, that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\left(-j_\ell\left(k\,r\right)\,\frac{\mathrm{d}}{\mathrm{d}r}y_\ell\left(k\,r\right) + y_\ell\left(kr\right)\,\frac{\mathrm{d}}{\mathrm{d}r}j_\ell\left(k\,r\right)\right)\right] = 0\,, \tag{2}$$

and so the Wronskian can be evaluated for any r. Hint: You obtain three terms by differentiation, *(i)* by the prefactor $r^2 \to 2\,r$, *(ii)* a mixed term, vanishes, and *(iii)* a term which corresponds to the double differentiations of the Bessel functions.

Task 3 (50 points)
You are given the current distribution

$$\vec{J}_0(\vec{r}) = J_{0,z}(\vec{r})\,\hat{\mathrm{e}}_z = \frac{I_0}{a^2}\,\frac{z}{a}\,\exp\left(-\frac{r^2}{a^2}\right)\,\hat{\mathrm{e}}_z\,, \tag{3}$$

where I_0 is a constant (it has the physical dimension of a current), and $r = \sqrt{x^2+y^2+z^2}$.

(a) Forget about vector spherical harmonics, for the moment. Carry out a multipole decomposition of the z components $J_{0,z}(\vec{r})$ and show that only the $\ell = 1$, $m = 0$ component is nonvanishing. (You may assume that the spherical harmonics form a complex basis of angular functions.)

(b) Use the multipole decomposition in the form

$$\vec{A}_0\left(\vec{r}\right) \approx \frac{\mathrm{i}k}{\epsilon_0\,c^2}\,\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{k^\ell\,h^{(1)}_\ell\left(k\,r\right)}{(2\ell+1)!!}\,Y_{\ell m}\left(\theta,\varphi\right)\int \vec{J}_0\left(\vec{r}'\right)\,r'^{\ell+2}\,Y^*_{\ell m}\left(\theta',\varphi'\right)\mathrm{d}r'\,\mathrm{d}\Omega'\,, \tag{4}$$

in order to evaluate the vector potential. Use the exact expression for the Hankel function $h^{(1)}_{\ell=1}\left(k\,r\right)$.

(c) Calculate the limiting forms in the radiation zone ($k\,r \gg 1$), and in the near-field zone ($k\,r \ll 1$).

The tasks are due Tuesday, 31–OCT–2023.