Task 1 (40 extra points)
Show that the Coulomb-gauge scalar potential, written as a retarded integral respecting causality,

$$\Phi_C(\vec{r}, t) = \int \mathrm{d}t' \int \mathrm{d}^3 r' \, G_R(\vec{r} - \vec{r}', t - t') \left(\frac{1}{c^2} \frac{\partial}{\partial t'} \mathcal{J}(\vec{r}', t') + \rho(\vec{r}', t') \right) , \tag{1}$$

with conventions as outlined in the lecture notes, fulfills the auxiliary condition for the scalar potential and longitudinal current in Coulomb gauge,

$$\epsilon_0 \frac{\partial}{\partial t} \vec{\nabla} \Phi_C \left(\vec{r}, t\right) = \vec{J}_{\parallel} \left(\vec{r}, t\right) , \tag{2}$$

as well as the equation

$$\vec{\nabla}^2 \Phi_C \left(\vec{r}, t\right) = -\frac{1}{\epsilon_0} \rho \left(\vec{r}, t\right) \, . \tag{3}$$

You may use lecture notes, but you should explain every step in your derivation in your own words and, if necessary, supplement intermediate steps not explicitly written in the notes.

The tasks are due Tuesday, 31–OCT–2023.