Task 1 (30 points)
Write (for yourself) a "cheat sheet" that summarizes essential formulas pertaining to ordinary and spherical Bessel functions (denoted as $J_n(x)$ and $j_n(x)$, respectively). The cheat sheet should contain

- defining differential equations,
- integral representations,
- asymptotics for small argument,
- asymptotics for large argument.

Then, write a computer program that plots $J_n(x)$ and $j_n(x)$ for $n = 0, 1, 2, 3$, over relevant range of the argument x (say, $0 < x < 20$). Convince yourself that only Bessel functions of order $n = 0$ are nonvanishing at the origin ($x = 0$). For unspeciifed extra credit, repeat the task for the ordinary and spherical Neumann functions (denoted as $Y_n(x)$ and $y_n(x)$).

Task 2 (20 points)
Define the Hankel function as

$$h_\ell^{(1)}(\rho) = j_\ell(\rho) + \mathrm{i}\, y_\ell(\rho) \,, \tag{1}$$

and show that

$$\mathrm{i}\, k\, j_\ell(k\, r_<)\; h_\ell^{(1)}(k\, r_>) \;\longrightarrow\; \frac{1}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}} \tag{2}$$

in the limit $k \to 0$, both by looking up the asymptotics as well as by considering the explicit form of the first few functions, say, for $\ell = 0, 1, 2, 3$.

Task 3 (20 points)
Tabulate the first few Hankel functions and convince yourself, by looking up the constituent j_ℓ and y_ℓ, that the Hankel functions can be written in terms of $\exp(\mathrm{i}\rho)$ and powers of ρ. Show that

$$h_0^{(1)}(\rho) = -\mathrm{i}\,\frac{\mathrm{e}^{\mathrm{i}\rho}}{\rho}\,, \qquad h_1^{(1)}(\rho) = -\frac{\mathrm{e}^{\mathrm{i}\rho}}{\rho} - \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\rho}}{\rho^2}\,, \qquad h_2^{(1)}(\rho) = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\rho}}{\rho} - \frac{3\mathrm{e}^{\mathrm{i}\rho}}{\rho^2} - \frac{3\mathrm{i}\mathrm{e}^{\mathrm{i}\rho}}{\rho^3}\,. \tag{3}$$

Show, by an explicit calculation for the three example cases above, that as $\rho \to +\infty$,

$$h_\ell^{(1)}(\rho) \to -\mathrm{i}\,\frac{\mathrm{e}^{\mathrm{i}(\rho-\ell\pi/2)}}{\rho} = (-\mathrm{i})^{\ell+1}\,\frac{\mathrm{e}^{\mathrm{i}\rho}}{\rho}\,, \qquad \mathrm{e}^{\mathrm{i}(-\ell\pi/2)} = \left(\mathrm{e}^{-\mathrm{i}\pi/2}\right)^\ell = (-\mathrm{i})^\ell\,, \qquad \rho \to \infty\,, \tag{4}$$

i.e., that the Hankel functions of different order only differ by a complex phase in the limit $\rho \to +\infty$.
Then, repeat the above exercise for the Hankel functions of the second kind, $h_0^{(2)}(\rho)$, $h_1^{(2)}(\rho)$, $h_2^{(2)}(\rho)$.

Task 4 (20 points) For a harmonically oscillating current density $\vec{J}(\vec{r}\,', t) = \vec{J}_0(\vec{r}\,') \exp(-\mathrm{i}\omega_0 t)$, of frequency ω_0, show that

$$\vec{A}(\vec{r}, t) = \mathrm{e}^{-\mathrm{i}\omega_0 t} \int \mathrm{d}^3 r'\; \vec{J}_0(\vec{r}\,')\, \frac{\exp(\mathrm{i}\omega_0 |\vec{r} - \vec{r}\,'|/c)}{4\pi\epsilon_0\, c^2\, |\vec{r} - \vec{r}\,'|}\,, \tag{5}$$

by first Fourier transforming the retarded Green function to ("mixed") frequency-coordinate space, applying it to a function which has only got a single Fourier component (i.e., describes a "monochromatic" single-frequency current density), and then Fourier backtransforming to space-time. The way in which you do the calculation is important. Hint: How would $\vec{J}(\vec{r}\,', \omega)$ look in frequency space if it has only a single frequency component at $\omega = \omega_0$?

The tasks are due Tuesday, 31–OCT–2023.