Task 1 (50 points)
Consider the expressions (with the symbols explained as given in the lecture)

$$\Psi(\vec{r},t) = \Psi_{\rm hom}(\vec{r},t) + \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3 r' \, \frac{1}{|\vec{r}-\vec{r}'|} \, F\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right) , \tag{1}$$

as well as

$$\Psi(\vec{r},t) = \Psi_{\rm hom}(\vec{r},t) + \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3 r' \, \frac{1}{|\vec{r}-\vec{r}'|} \, F\left(\vec{r}', t + \frac{|\vec{r}-\vec{r}'|}{c}\right) . \tag{2}$$

Verify by direct calculation that these fulfill the relation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right) \Psi(\vec{r},t) = \frac{1}{\epsilon_0} \, F(\vec{r},t) \, . \tag{3}$$

By "showing explicitly", it is understood that you carry out the differentiations with respect to time and space explicitly, i.e., by directly acting on the expression

$$\int \mathrm{d}^3 r' \, \frac{1}{|\vec{r}-\vec{r}'|} \, F\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right) , \tag{4}$$

with differentiations with respect to t and $\vec{r}$. The resulting expressions will contain a number of derivatives and gradients. Please do not forget that the second argument of the F function, $t - \frac{|\vec{r}-\vec{r}'|}{c}$, also depends on $\vec{r}$.

Task 3 (100 points = five times 20 points)

We investigate the harmonic oscillator using Green function techniques and apply the Green function formalism to the harmonic oscillator. The "displacement" $x = x(t)$ of a damped, harmonic oscillator with unit mass $m = 1$ satisfies the equation

$$\ddot{x} + \gamma \, \dot{x} + \omega_0^2 \, x = f(t) \, . \tag{5}$$

(a) Obtain the Green function $g = g(t-t')$ for this equation. Note that it has to fulfill the equation

$$\ddot{g}(t-t') + \gamma \, \dot{g}(t-t') + \omega_0^2 \, g(t-t') = \delta(t-t') \, . \tag{6}$$

(b) Why is this Green function naturally obtained as the retarded Green function? Why do you have to change the sign of the damping term in order to obtain the advanced Green function? Consider the effect of time reversal on the equation of motion, and provide an illustrative discussion.

(c) Assume that the "force" is given by $f(t) = v_0 \, \delta(t)$, where v_0 has the dimension of a velocity. Under the boundary conditions $x(-\infty) = 0$, and $\dot{x}(-\infty) = 0$, evaluate $x(t)$.

(d) Assume that the "force" is given by $f(t) = f_0 \left[\Theta(t) + \Theta(-t) \exp(t/\tau)\right]$, with vanishing boundary conditions in the infinite past, $x(-\infty) = 0$, and $\dot{x}(-\infty) = 0$. Possibly with the help of computer algebra, evaluate $x(t)$.

(e) Evaluate the work done by the driving force on the oscillator (per unit mass) in the time interval from $-\infty$ to $+\infty$ as a function of ω_0, γ, and τ.

Now let $\gamma = \frac{1}{10}\omega_0$. Plot the work done by the driving force divided by the final (potential) energy stored in the oscillator as a function of $L = \ln(\omega_0 \tau)$ from $L = -4$ to $L = 4$.

You may utilize the hints given during the lecture, but please formulate your thoughts so that it becomes clear that you show all your work.

The tasks are due Thursday, 19–OCT–2023, with a possible extension to Monday, 23–OCT–2023.