Task 1 (30 points)
You are given the electric field and current

$$\vec{E}(\vec{r},t) = E_0 \, \exp\left[-\frac{(x-ct)^2}{2L^2}\right] \cos\left(k_0\,(x-ct)\right) \hat{e}_z \,, \qquad \vec{J}\,(\vec{r},t) = \vec{0}\,. \tag{1}$$

The electric field describes a wave pulse of an electromagnetic wave, z polarized, that travels in the positive x direction.
(a) Show that $\vec{\nabla}\cdot\vec{E}(\vec{r},t) = 0$. **(b)** Calculate $\rho(\vec{r},t)$. **(c)** Using Faraday's law, $\vec{\nabla}\times\vec{E}(\vec{r},t) + \frac{\partial}{\partial t}\vec{B}(\vec{r},t) = 0$, show that $\vec{B}(\vec{r},t)$ is given by

$$\vec{B}(\vec{r},t) = -\frac{E_0}{c} \, \exp\left[-\frac{(x-ct)^2}{2L^2}\right] \cos\left(k_0\,(x-ct)\right) \hat{e}_y \,. \tag{2}$$

(d) Show that the Ampere–Maxwell law is fulfilled, $\vec{\nabla}\times\vec{B}(\vec{r},t) - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{r},t) = \vec{0}$. **(e)** Make a plot of the z component $E_z(\vec{r},t)$ of the electric field $\vec{E}(\vec{r},t) = E_z(\vec{r},t)\,\hat{e}_z$ given in Eq. (1) for suitable parameters. I.e., interpret

$$f(x,t) = \frac{E_z(x,t)}{E_0} = \exp\left[-\frac{(x-ct)^2}{2L^2}\right] \cos\left(k_0\,(x-ct)\right) \tag{3}$$

as a function of x, for given t. Choose at least two different values for t, and describe how the wave packet evolves. Choose a plotting software of your choice. Possible values include $t = t_1 = 0$ and $t = t_2 = 3.0\times10^{-11}\,\mathrm{s}$, $L = 0.005\,\mathrm{m}$, and $k_0 = 10^4\,\mathrm{m}^{-1}$. However, it is your choice to pick a suitable range of values, so that the shape of the wave pulse becomes visible.

Task 3 (30 points)
Starting from the momentum-space (Fourier transform) expressions, derive the expression

$$\begin{aligned} G_A\left(\vec{r}-\vec{r}\,',t-t'\right) &= \frac{c}{4\pi\epsilon_0}\,\frac{1}{|\vec{r}-\vec{r}\,'|}\,\Theta(t'-t)\left\{\delta\Big(|\vec{r}-\vec{r}\,'| - c\,(t'-t)\Big) - \delta\Big(|\vec{r}-\vec{r}\,'| + c\,(t'-t)\Big)\right\} \\ &= \frac{c}{2\pi\epsilon_0}\,\Theta(t'-t)\,\delta\Big((\vec{r}-\vec{r}\,')^2 - c^2\,(t-t')^2\Big) \end{aligned} \tag{4}$$

for the advanced Green function, by following the integrations given in the lecture for the retarded Green function, and using the "advanced" integration contour.

Task 2 (30 extra points)
Show that the Fourier backtransformation of the Feynman Green function is given by

$$\begin{aligned} G_F(\vec{r}-\vec{r}\,',\tau) &= \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\,\frac{\exp\left(\mathrm{i}\,\sqrt{\omega^2+\mathrm{i}\epsilon}\,\frac{|\vec{r}-\vec{r}\,'|}{c} - \mathrm{i}\omega\tau\right)}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|} \\ &= -\,\mathrm{i}\,\frac{c}{4\pi^2\epsilon_0}\,\frac{1}{c^2\,\tau^2 - (\vec{r}-\vec{r}\,')^2 - \mathrm{i}\,\epsilon}\,, \end{aligned} \tag{5}$$

where you assume that the branch cut of the square root function is chosen so that $\mathrm{Im}\sqrt{\omega^2+\mathrm{i}\epsilon} > 0$ throughout the entire complex plane. Verify and explain, in your own words, why this implies that the branch cut of the square root function should be positioned along the positive real axis.

The tasks are due Thursday, 19–OCT–2023.