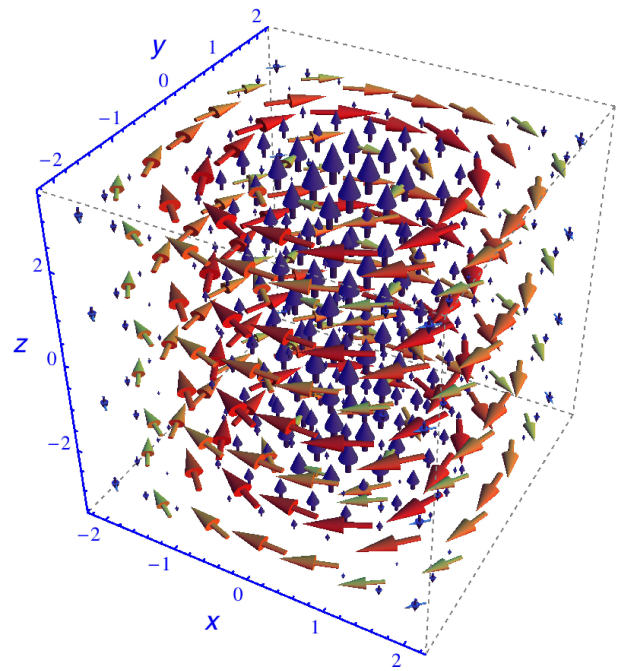
Electrodynamics II — Physics/6211

Missouri S & T (Rolla, Missouri)



Dr. Ulrich Jentschura
Course Instructor
Physics Building Room #205
1315 North Pine Street
Rolla, MO 65409–0640
e-mail ulj@mst.edu

Contents

Chapter 1

From the Maxwell Equations To Waves

1.1 Orientation

In the second series of lectures of electrodynamics, we study the dynamical aspects of the physics implied by the Maxwell equations and the Lorentz force law.

An outline of the topics can be given as follows:

We first investigate some general properties of the Maxwell equations. Symmetry properties and conservation laws for the fields will be examined. The solutions of the equations in a uniform isotropic medium will be developed using scalar and vector potentials. Finally, the wave equation will be obtained for a uniform, isotropic medium. This is the subject of the current Chapter 1. Also, we shall see how the complex formalism, with complex phases $\exp(-\mathrm{i}\,\omega\,t)$, naturally emerges if one tries to describe dissipation and energy storage for electromagnetic waves in continuous-wave mode.

The famous Maxwell equations which describe electromagnetic radiation look different if one takes the response of the medium into account; they then become the macroscopic, or phenomenological, Maxwell equations. The phenomenological modifications summarize the response of the medium.

In Chapter 2, the Green functions for the wave equation are discussed. We shall find that the scalar and vector potentials of electrodynamics are coupled to the sources (charge and current densities) via equations which have the structure of wave equations (the precise form depends on the gauge). Therefore, it is of prime importance to study the Green functions of electromagnetic radiation, in order to be able to integrate the wave equations and to calculate the electromagnetic radiation pattern emitted from oscillatory sources.

We investigate electromagnetic radiation in Chapter 3, based on the Helmholtz equation which is obtained from the wave equation by Fourier transformation with respect to time (but still in coordinate space). Multipole corrections to radiation are obtained. The radiation by simple systems includes oscillating electric and magnetic dipoles, and quadrupole fields. Potentials due to moving charges are also discussed.

The discussion of waveguides in Chapter 4 includes TE and TM modes, and illustrative examples, and graphical representations of the oscillating fields in the wave guide. Rectangular and cylindrical configurations are discussed. Finally, the Casimir effect, which results in field configurations with boundary conditions, together with the attractive Casimir force between plates, is derived. The latter topic is somewhat beyond the scope of the current course and requires the quantization of the electromagnetic field. It is still being included as the quantization is required on a ruimentary level only, and because the effect can be derived in a transparent calculation.

A discussion of waves in media follows in Chapter 5. Phase and group velocity are distinguished. The Kramers–Kronig relations fulfilled by the frequency-dependent dielectric "constant" of the medium are discussed. It is shown that the analytic properties of the relative permittivity ϵ_r allow us to formulate sum rules that isolate specific asymptotic limits in the functional form of the relative permittivity. Discussions of the Clausius–Mosotti equation, and of the Lyddane-Sachs-Teller equations, complement the discussion.

1.2 Time–Dependent Electromagnetic Fields

1.2.1 Maxwell Equations in Materials

Our starting point will be the experimentally deduced Maxwell equations. In vacuum, they read

$$\boxed{\text{Gauss's Law:}} \quad \vec{\nabla} \cdot \vec{E}\,(\vec{r}, t) = \frac{1}{\epsilon_0}\,\rho\,(\vec{r}, t)\,, \tag{1.1a}$$

$$\boxed{\text{Absence of Magnetic Monopoles:}} \quad \vec{\nabla} \cdot \vec{B}\,(\vec{r}, t) = 0\,, \tag{1.1b}$$

$$\boxed{\text{Faraday's Law:}} \quad \vec{\nabla} \times \vec{E}\,(\vec{r}, t) = -\frac{\partial}{\partial t}\vec{B}\,(\vec{r}, t)\,, \tag{1.1c}$$

$$\boxed{\text{Ampere-Maxwell Law:}} \quad \vec{\nabla} \times \vec{B}\,(\vec{r}, t) = \frac{1}{c^2}\,\frac{\partial}{\partial t}\vec{E}\,(\vec{r}, t) + \mu_0\,\vec{J}\,(\vec{r}, t) \tag{1.1d}$$

formulated in terms of the electric field $\vec{E}\,(\vec{r}, t)$ and of the magnetic induction $\vec{B}\,(\vec{r}, t)$. Equations (1.1b) and (1.1c) are called homogeneous equations, because they do not couple the fields to their sources, but only the fields to each other. Equations (1.1a) and (1.1d) are the inhomogeneous equations and couple the fields to the charge density $\rho\,(\vec{r}, t)$ and to the current density $\vec{J_0}\,(\vec{r}, t)$.

The two equations that couple the fields to the sources, Eqs. (1.1a) and (1.1d), are actually not independent of each other. Let us take the divergence of Eq. (1.1d),

$$\begin{aligned}
\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B}\,(\vec{r}, t)\right) &= \frac{1}{c^2}\,\frac{\partial}{\partial t}\vec{\nabla} \cdot \vec{E}\,(\vec{r}, t) + \mu_0 \vec{\nabla} \cdot \vec{J}\,(\vec{r}, t) \\
&= \frac{1}{c^2}\,\frac{\partial}{\partial t}\vec{\nabla} \cdot \vec{E}\,(\vec{r}, t) - \mu_0\,\frac{\partial}{\partial t}\rho\,(\vec{r}, t) \\
&= \frac{1}{c^2}\,\frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{E}\,(\vec{r}, t) - \frac{1}{\epsilon_0}\,\rho\,(\vec{r}, t)\right)\,,
\end{aligned} \tag{1.2}$$

where we have used the identity $\mu_0\,\epsilon_0 = 1/c^2$ and the charge conservation condition

$$\vec{\nabla} \cdot \vec{J}(\vec{r}, t) + \frac{\partial}{\partial t}\rho(\vec{r}, t) = 0\,. \tag{1.3}$$

The divergence of Eq. (1.1d) thus is equivalent to the time derivative of Eq. (1.1b). In other words, we can exchange Gauss's law given in Eq. (1.1a) for the combined statement of charge conservation and an initial condition for the electric field determined by the condition $\vec{\nabla} \cdot \vec{E}\,(\vec{r}, t_0) = \rho\,(\vec{r}, t_0)\,/\epsilon_0$ at a particular start time t_0.

If we insert a medium, then the constituent particles in the medium will rearrange as a result of the externally applied free charge density $\rho_0\,(\vec{r}, t)$ (which ignores the induced charge density in the medium) and the free current density $\vec{J}_0\,(\vec{r},t)$ (which ignores the currents induced in the medium), and we have

$$\rho(\vec{r},t) = \rho_0(\vec{r},t) + \rho_{\mathrm{mat}}(\vec{r},t)\,, \tag{1.4}$$

$$\vec{J}(\vec{r},t) = \vec{J}_0(\vec{r},t) + \vec{J}_{\mathrm{mat}}(\vec{r},t)\,. \tag{1.5}$$

Here, the additional terms are due to the material charge density $\rho_{\mathrm{mat}}\,(\vec{r},t)$ and the material current density $\vec{J}_{\mathrm{mat}}\,(\vec{r},t)$ which refer to the induced quantities in the medium.

In that case, we can write (of course) the inhomogeneous Maxwell equations as

$$\vec{\nabla}\cdot\vec{E}(\vec{r},t) = \frac{1}{\epsilon_0}\,\left[\rho_0(\vec{r},t) + \rho_{\mathrm{mat}}(\vec{r},t)\right]\,, \tag{1.6a}$$

$$\vec{\nabla}\times\vec{B}(\vec{r},t) = \frac{1}{c^2}\,\frac{\partial}{\partial t}\vec{E}(\vec{r},t) + \mu_0\,\left[\vec{J}_0(\vec{r},t) + \vec{J}_{\mathrm{mat}}(\vec{r},t)\right]\,. \tag{1.6b}$$

The inhomogenenous Maxwell equations are valid everywhere inside the medium. However, we could also try to introduce the displacement field $\vec{D}\,(\vec{r},t)$ and the magnetic field $\vec{H}\,(\vec{r},t)$, and to define these fields in such a way that they couple only to the free, measurable fields,

$$\vec{\nabla}\cdot\vec{D}\,(\vec{r},t) = \rho_0\,(\vec{r},t)\,, \tag{1.7a}$$

$$\vec{\nabla}\times\vec{H}\,(\vec{r},t) = \frac{\partial}{\partial t}\vec{D}\,(\vec{r},t) + \vec{J}_0\,(\vec{r},t)\,. \tag{1.7b}$$

In this form, the Maxwell equations are not useful from a fundamental physics point of view, but very useful from a practical point of view. The problem then is how to define the displacement and magnetic fields so that Eq. (1.7) is compatible with Eq. (1.6).

Before we indulge in this endeavour, we list once more the four differential equations, fulfilled by the fields in the presence of sources, which in SI mksA units read as follows,

Gauss's Law (inhomogeneous): $$\vec{\nabla}\cdot\vec{D}\,(\vec{r},t) = \rho_0\,(\vec{r},t)\,, \tag{1.8a}$$

Absence of Magn. Monopoles (homogeneous): $$\vec{\nabla}\cdot\vec{B}\,(\vec{r},t) = 0\,, \tag{1.8b}$$

Faraday's Law (homogeneous): $$\vec{\nabla}\times\vec{E}\,(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}\,(\vec{r},t)\,, \tag{1.8c}$$

Ampere-Maxwell Law (inhomogeneous): $$\vec{\nabla}\times\vec{H}\,(\vec{r},t) = \frac{\partial}{\partial t}\vec{D}\,(\vec{r},t) + \vec{J}_0\,(\vec{r},t) \tag{1.8d}$$

for the displacement field $\vec{D}\,(\vec{r},t)$, the magnetic induction field $\vec{B}\,(\vec{r},t)$, the electric field $\vec{E}\,(\vec{r},t)$, and the magnetic field $\vec{H}\,(\vec{r},t)$. The sources of the fields are the free charge density $\rho_0\,(\vec{r},t)$ (which ignores the induced charge density in the medium) and the free current density $\vec{J}_0\,(\vec{r},t)$ (which ignores the currents induced in the medium). The inhomogeneous Maxwell equations involve the phenomenologically induced $\vec{D}$ and $\vec{H}$ fields, whereas the homogeneous equations involve the $\vec{E}$ and $\vec{B}$ fields.

[Optional remarks: The homogeneous Maxwell equations do not change in the presence of a medium. The deeper reason can be seen when one formulates the electromagnetic theory on manifolds, where the two homogeneous equations take the form $\mathrm{d}F = 0$, where d is this case stands for the "outer differential" of the "two-form" $F = \mathrm{d}A$, where A is the "one-form" describing the vector potential. The two homogeneous Maxwell equations then read as $\mathrm{d}^2 A = 0$, which is a fundamental property of differential forms. At a much later stage, you might come across the fact that the Maxwell equations actually need to be supplemented by so-called quantum corrections, which add specific terms to the Maxwell equations (so-called "Maxwell Lagrangian"). These added terms only modify the two inhomogeneous equations by higher powers of the

Table 1.1: Physical quantities important for electrodynamics are their physical units in the SI mksA unit system.

Physical Quantity	SI Unit
$\vec{B}$	1 Tesla$= 1\mathrm{V\,s\,m^{-2}}$
$\vec{H}$	$\mathrm{A\,m^{-1}}$
$\vec{M}$	$\mathrm{A\,m^{-1}}$
$\vec{E}$	$\mathrm{V\,m^{-1}}$
$\vec{D}$	$\mathrm{C\,m^{-2}}$
$\vec{P}$	$\mathrm{C\,m^{-2}}$
ϵ_0	$\mathrm{A\,s\,V^{-1}\,m^{-1}}$
ρ_0	$\mathrm{C\,m^{-3}}$
$\vec{J}_0$	$\mathrm{A\,m^{-2}}$

electric and magnetic fields; the speed of light changes under the presence of very strong fields (e.g., within magnetars or neutron stars), and this is described by the modification of the inhomogeneous equations. The homogeneous equations remain untouched.]

This set of equations requires information concerning the properties and responses of the materials in the region of the fields. These are isolated in the constituent equations

$$\vec{D}\left(\vec{r},t\right) = \epsilon_0\,\vec{E}\left(\vec{r},t\right) + \vec{P}\left(\vec{r},t\right)\ , \tag{1.9}$$

$$\vec{H}\left(\vec{r},t\right) = \frac{1}{\mu_0}\,\vec{B}\left(\vec{r},t\right) - \vec{M}\left(\vec{r},t\right)\ , \tag{1.10}$$

where $\vec{P}\left(\vec{r},t\right)$ is the polarization field and $\vec{M}\left(\vec{r},t\right)$ is the magnetization field for the materials.

From

$$\epsilon_0\,\vec{\nabla}\cdot\vec{E}\left(\vec{r},t\right) = \rho_0(\vec{r},t) + \rho_{\mathrm{mat}}(\vec{r},t) \tag{1.11}$$

and thus

$$\vec{\nabla}\cdot\vec{D}\left(\vec{r},t\right) = \epsilon_0\,\vec{\nabla}\cdot\vec{E}\left(\vec{r},t\right) + \vec{\nabla}\cdot\vec{P}\left(\vec{r},t\right) = \left\{\rho_0(\vec{r},t) + \rho_{\mathrm{mat}}(\vec{r},t)\right\} - \rho_{\mathrm{mat}}(\vec{r},t)\,, \tag{1.12}$$

we can infer that

$$\vec{\nabla}\cdot\vec{P}\left(\vec{r},t\right) = -\rho_{\mathrm{mat}}\left(\vec{r},t\right) = -\rho_p\left(\vec{r},t\right) \tag{1.13}$$

where $\rho_p\left(\vec{r},t\right)$ is the polarization charge density. We can thus identify the material charge density ρ_{mat} as being exclusively equal to the polarization charge density $\rho_p\left(\vec{r},t\right)$ (no additional contributions from magnetic effects). The magnetization field $\vec{M}\left(\vec{r},t\right)$ couples to the magnetization current density $\vec{J}_{\mathrm{mag}}\left(\vec{r},t\right)$ as

$$\vec{\nabla}\times\vec{M}\left(\vec{r},t\right) = \vec{J}_{\mathrm{mag}}\left(\vec{r},t\right)\ . \tag{1.14}$$

The units of these quantities are summarized in Table 1.1 [see also Eq. (1.25)].

If Maxwell's equations are written with the charge densities and current densities as parameters, we obtain the *microscopic* form for the equations, which reads (for convenience, we repeat all equations once more)

$$\vec{\nabla} \cdot \vec{E}(\vec{r},t) = \frac{1}{\epsilon_0}\left[\rho_0(\vec{r},t) + \rho_{\rm mat}(\vec{r},t)\right], \tag{1.15a}$$

$$\vec{\nabla} \cdot \vec{B}(\vec{r},t) = 0, \tag{1.15b}$$

$$\vec{\nabla} \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t), \tag{1.15c}$$

$$\vec{\nabla} \times \vec{B}(\vec{r},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{r},t) + \mu_0\left[\vec{J}_0(\vec{r},t) + \vec{J}_{\rm mat}(\vec{r},t)\right]. \tag{1.15d}$$

In these equations, $\rho_{\rm mat}(\vec{r},t)$ and $\vec{J}_{\rm mat}(\vec{r},t)$ are the charge density and current density in the materials, exclusive of the free charge and current densities ρ_0 and $\vec{J}_0$. The charge density and current density in the materials are due to a rearrangement of the constituent atoms or molecules in the material, as a reaction to the free charge and current densities ρ_0 and $\vec{J}_0$ which have been brought in to the medium, externally. These exist in response to the externally applied fields.

If we assume the externally applied charges and currents to propagate independently of the internally induced quantities, then charge conservation has to hold independently for both sets of quantities. From the conservation of charge, which implies that for the free charge density and current,

$$\vec{\nabla} \cdot \vec{J}_0 = -\frac{\partial}{\partial t}\rho_0, \tag{1.16}$$

we deduce that

$$\vec{\nabla} \cdot \vec{J}_{\rm mat} = -\frac{\partial}{\partial t}\rho_p = \frac{\partial}{\partial t}\left[\vec{\nabla} \cdot \vec{P}(\vec{r},t)\right] = \vec{\nabla} \cdot \left\{\frac{\partial}{\partial t}\vec{P}(\vec{r},t)\right\}. \tag{1.17}$$

We cannot deduce from this consideration that

$$\vec{J}_{\rm mat} \neq \left\{\frac{\partial}{\partial t}\vec{P}(\vec{r},t)\right\}, \tag{1.18}$$

because there might be additional magnetically induced effects. (The divergence of any curl of a vector field vanishes.) However, we can define the

$$\boxed{\textbf{Polarization Current:}} \qquad \vec{J}_p(\vec{r},t) = \frac{\partial}{\partial t}\vec{P}(\vec{r},t) \tag{1.19}$$

to be the current density associated with the polarization charge density, given by the moving induced charges. The physical mechanism is that the externally applied electric field polarizes the medium. As it changes, the polarization changes, and for it to change, charged particles associated with the polarization charge density have to move. This gives rise to the polarization current $\vec{J}_p$.

The current density in the material $\vec{J}_{\rm mat}$ is the sum of the polarization current density $\vec{J}_p$ and a magnetically induced term. Formally, we first observe that we can rewrite the quantity $\vec{J}_{\rm mat}$ as it appears in the Ampere-Maxwell law as the sum of two terms, the first of which has zero divergence, the second of which has zero curl:

$$\vec{J}_{\rm mat} = \left[\vec{J}_{\rm mat} - \vec{J}_p\right] + \vec{J}_p. \tag{1.20}$$

Charge conservation holds separately for the free and the "material" current and charge densities,

$$\boxed{\textbf{Charge Conservation/Free Current:}} \qquad \vec{\nabla} \cdot \vec{J}_0 + \frac{\partial}{\partial t}\rho_0 = 0, \tag{1.21}$$

and

$$\boxed{\textbf{Charge Conservation/Material Current:}} \qquad \vec{\nabla} \cdot \vec{J}_{\rm mat} + \frac{\partial}{\partial t}\rho_{\rm mat} = 0, \tag{1.22}$$

as well as for the polarization charge current density and the corresponding charge density,

$$\boxed{\textbf{Charge Conservation/Polarization Current:}} \qquad \vec{\nabla} \cdot \vec{J}_p + \frac{\partial}{\partial t} \rho_p = 0 \,. \tag{1.23}$$

From $\rho_p = \rho_{\rm mat}$, we can deduce that

$$\vec{\nabla} \cdot \left[\vec{J}_{\rm mat} - \vec{J}_p \right] = 0 \,. \tag{1.24}$$

Because its divergence vanishes, the difference of the current densities $\vec{J}_{\rm mat} - \vec{J}_p$ can be written as the curl of a vector field (magnetization field) $\vec{M}\,(\vec{r},t)$, i.e., as the magnetization current density $\vec{J}_m$ [see Eq. (1.14)],

$$\vec{J}_{\rm mag} = \vec{J}_{\rm mat} - \vec{J}_p = \vec{\nabla} \times \vec{M}\,(\vec{r},t) \;. \tag{1.25}$$

Here, $\vec{M}\,(\vec{r},t)$ is identified as the magnetization of the materials. In terms of $\vec{M}$ and $\vec{P}$, the Ampere-Maxwell law is given as follows (SI mksA units, and we use $\epsilon_0\,\mu_0 = c^{-2}$):

$$\begin{aligned}
\vec{\nabla} \times \vec{B}\,(\vec{r},t) &= \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}\,(\vec{r},t) + \mu_0\,\vec{J}_0\,(\vec{r},t) + \mu_0\,\frac{\partial}{\partial t}\vec{P}\,(\vec{r},t) + \mu_0\,\vec{\nabla} \times \vec{M}\,(\vec{r},t) \;, \\
\Rightarrow\; \vec{\nabla} \times \left[\vec{B}\,(\vec{r},t) - \mu_0\,\vec{M}\,(\vec{r},t)\right] &= \frac{1}{c^2}\frac{\partial}{\partial t}\left[\vec{E}\,(\vec{r},t) + \frac{1}{\epsilon_0}\vec{P}\,(\vec{r},t)\right] + \mu_0\,\vec{J}_0\,(\vec{r},t) \;, \\
\Rightarrow\; \vec{\nabla} \times \left[\frac{1}{\mu_0}\vec{B}\,(\vec{r},t) - \vec{M}\,(\vec{r},t)\right] &= \frac{\partial}{\partial t}\left[\epsilon_0\,\vec{E}\,(\vec{r},t) + \vec{P}\,(\vec{r},t)\right] + \vec{J}_0\,(\vec{r},t) \;, \\
\Rightarrow\; \vec{\nabla} \times \vec{H}\,(\vec{r},t) &= \frac{\partial}{\partial t}\vec{D}\,(\vec{r},t) + \vec{J}_0\,(\vec{r},t) \;.
\end{aligned} \tag{1.26}$$

This is consistent with the constituent equations. The Lorentz force then is

$$\boxed{\textbf{Lorentz Force:}} \qquad \vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \,. \tag{1.27}$$

We reemphasize that the Lorentz force cannot be deduced from the Maxwell equations and represents in some sense a "fifth Maxwell equation."

Remark: We have used the relation

$$\begin{aligned}
\mu_0 \vec{J} &= \mu_0 \vec{J}_0 + \mu_0 \vec{J}_p + \mu_0 (\vec{J}_{\rm mat} - \vec{J}_p) \\
&= \mu_0 \vec{J}_0 + \mu_0\,\frac{\partial \vec{P}}{\partial t} + \mu_0\,\vec{\nabla} \times \vec{M} \,.
\end{aligned} \tag{1.28}$$

1.2.2 Small Digression

In a typical dipole polarizable medium, the electric field is attenuated according to

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_r}\,\frac{1}{\epsilon_0}\,\rho_0 \,, \tag{1.29}$$

where, typically, $\epsilon_r > 1$, and ρ_0 is the free charge density. The electric field points from positive to negative charges, i.e., in the direction in which a fictitious positively charged test particle would move. The attenuation takes places because the molecules or atoms in the sample will arrange themselves such as to generate an electric field opposite to the one applied from outside.

Let us suppose that we have a uniform, isotropic material where the effect of the material can be described by a simple proportionality constant ("fudge factor") ϵ_r. Although the induced electric field is directed antiparallel to the externally applied electric field, the polarization field $\vec{P}$ is directed parallel to the externally applied field, i.e., against the direction in which a fictitious positively charged test particle would move due to the additionally induced electric field. Let us suppose that the uniform polarization field (its physical dimension is that of a dipole charge density, i.e., of a dipole moment per unit volume) can be written as

$$\vec{P} = (\epsilon_r - 1)\, \epsilon_0\, \vec{E}\,. \tag{1.30}$$

The field $\vec{P}$ is antiparallel to the electric field generated by the induced charges, and we have, locally,

$$\vec{\nabla} \cdot \vec{P} = -\rho_p\,. \tag{1.31}$$

Remark: One should appreciate a drawing where the elementary dipoles inside the material are directed with the dipole moment vectors pointing parallel to the external field, but the induced field pointing antiparallel to the external applied field.

Since ρ is a charge density, and since $\vec{\nabla}$ has physical dimension of inverse length, it is clear that $\vec{P}$ must have the physical dimension of a dipole density (dipole moment per volume). Also, locally, the divergence of the electric field is given by the total charge density

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\, (\rho_0 + \rho_p) = \frac{1}{\epsilon_0}\, \rho_0 - \frac{1}{\epsilon_0}\, \vec{\nabla} \cdot \vec{P}\,. \tag{1.32}$$

As usual, we identify

$$\vec{D} = \epsilon_0\, \vec{E} + \vec{P}\,, \qquad \vec{\nabla} \cdot \vec{D} = \rho_0\,, \tag{1.33}$$

which is equivalent to

$$\vec{\nabla} \cdot \left(\epsilon_0 \vec{E} + (\epsilon_r - 1) \epsilon_0 \vec{E} \right) = \rho_0\,, \tag{1.34}$$

and this in turn is equivalent to

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_r}\, \frac{1}{\epsilon_0}\, \rho_0\,. \tag{1.35}$$

The sign of the polarization $\vec{P}$ is determined by the fact that the induced dipoles in the material point, by definition, from the negative to the positive charges, and therefore, in a direction opposite to the additional electric field created by them. This direction of the dipole vector is already evident from the formula $\vec{p} = \int \vec{r}\, \rho(\vec{r})\, \mathrm{d}^3 r$, for the dipole moment, where $\vec{p}$ is the dipole moment vector, and regions with positive $\rho(\vec{r})$ count as positive contributions to the dipole moment.

In the most primitive, isotropic situations, we have

$$\vec{D} = \epsilon_r \epsilon_0\, \vec{E}\,. \tag{1.36}$$

In a non-isotropic medium, we have, by contrast,

$$D_i = \sum_{j=1}^{3} \epsilon_{ij}\, E_j = \epsilon_0\, E_i + \sum_{j=1}^{3} (\epsilon_{ij} - \epsilon_0\, \delta_{ij})\; E_j = \epsilon_0\, E_i + \sum_{j=1}^{3} \chi_{ij}\, E_j\,, \tag{1.37}$$

where the permittivity tensor ϵ_{ij} and the susceptibility tensor χ_{ij} are of rank two. If there are nonvanishing off-diagonal elements in this tensor, then it means that the polarization field is not necessarily parallel to the externally applied electric field.

This can happen, e.g., under the following circumstances: Let us suppose we have a regular arrangement of permanent dipoles that are free to move in the xy plane but cannot move into the z direction. In that case, if we apply an external field at an angle with the xy plane, then the internal dipoles will still arrange themselves as a reaction to the applied electric field, but they cannot move out of the xy plane, and therefore, they cannot align themselves directly opposite to the applied electric field. In that case, ϵ_{ij} will not be a fully diagonal tensor.

In Eq. (1.38), we have already encountered the necessary generalization for situations in which the response of the system is non-diagonal, non-local and possible, delayed,

$$P_i(\vec{r}, t) = \sum_{j=1}^{3} \int \chi_{ij}(\vec{r} - \vec{r}', t - t')\, E_j(\vec{r}', t')\, \mathrm{d}^3 r'\, \mathrm{d}t' \,. \tag{1.38}$$

Here, $\chi(\vec{r}, t)$ is the electric susceptibility tensor for the system. Equation (1.38) describes a delayed response of the system. If the electric field ramps up slowly, then the atoms of the sample (if they are polarizable) have 'no choice' but to align themselves such as to produce an electric field that is directed against the originally applied, external electric field $\vec{E}(\vec{r}', t')$. However, things change when the stimulating electric field has a nonvanishing frequency. In that case, we can invoke an analogy with a child on a swing. If we swing the child at a very low frequency, then the movement of the child will follow suit. However, when the frequency of the pushing is much higher than the resonant frequency of the swing, then a delayed response of the child may result, and there may be a phase difference between the oscillatory motion of the child and the phase of the swinging force, eventually resulting in a phase difference of $180°$ with respect to the oscillation (driven damped oscillator). In the frequency domain, and for a homogeneous sample, we have

$$P_i(\vec{r}, \omega) = \sum_{j=1}^{3} \int \mathrm{d}^3 r'\, \chi_{ij}(\vec{r} - \vec{r}', \omega)\, E_j(\vec{r}', \omega) \tag{1.39}$$

in the position-frequency domain (Fourier transform), or even

$$P_i(\vec{k}, \omega) = \sum_{j=1}^{3} \chi_{ij}(\vec{k}, \omega)\, E_j(\vec{k}, \omega) \tag{1.40}$$

in the wave-vector-frequency domain. This situation will be of concern for us in the following.

Let us consider a situation with oriented dipoles, with an electric field pointing in the positive x direction. The distance between the separated charges in the dipoles is Δz. The induced electric field points to the left.

Let the outer, non-compensated layers take up a distance Δz. There is a non-compensated layer on the "right", with positive charge density, and a non-compensated negative layer on the left, with negative charge density (see also Fig. 1.1).

The positively charged layer corresponds to a dipole density of

$$P = \frac{q_{\mathrm{ind}}\, \Delta z}{V} = \frac{q_{\mathrm{ind}}\, \Delta z}{A\, \Delta z} = \frac{q_{\mathrm{ind}}}{A} = \sigma_{\mathrm{ind}} \,. \tag{1.41}$$

The induced electric field points to the left and is

$$E_{\mathrm{ind}} = -\frac{\sigma_{\mathrm{ind}}}{2\epsilon_0} - \frac{\sigma_{\mathrm{ind}}}{2\epsilon_0} = -\frac{\sigma_{\mathrm{ind}}}{\epsilon_0} = -\frac{q_{\mathrm{ind}}}{A\, \epsilon_0} \,. \tag{1.42}$$

We have shown the identity

$$-\epsilon_0\, E_{\mathrm{ind}} = \frac{q}{A} = P \,, \qquad -\epsilon_0\, \vec{E}_{\mathrm{ind}} = \vec{P} \,. \tag{1.43}$$

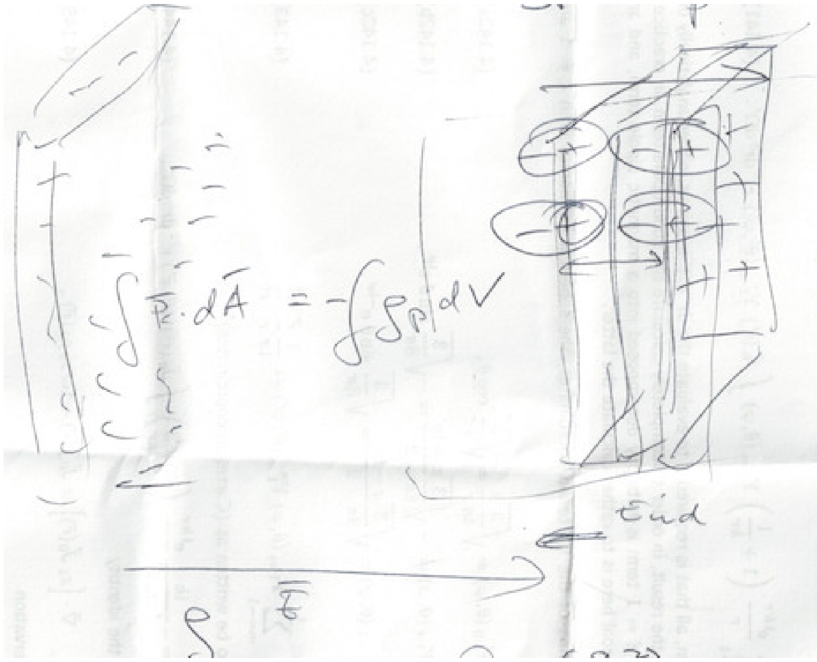


Figure 1.1: A depiction of oriented dipoles and the induced electric field, and charge density. The explanation is in the text.

Whatever is left when we subtract the induced electric field must be due to the free charge density:

$$\vec{\nabla} \cdot (\vec{E} - \vec{E}_{\text{ind}}) = \frac{1}{\epsilon_0} \rho_0 \tag{1.44}$$

or

$$\vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_0 \, . \tag{1.45}$$

This shows once more the consistency of the formalism. At some risk to oversimplification, we can say that the induced field $\vec{E}_{\text{ind}}$ and the free charge density ρ_0 counteract: The larger the modulus $|\vec{E}_{\text{ind}}|$, i.e., the more polarizable the medium is, the more free charge density one needs in order to maintain a given field $\vec{E}$.

Example: The case of a "perfectly polarizable medium" corresponds to the limit $\epsilon_r \to \infty$. In this case, one has $\vec{E} \to \vec{0}$ (total field inside the material is compensated and is zero) while $\vec{E}_{\text{ind}} \neq \vec{0}$, as the induced field has to compensate the external field E_{ext} inside the material. Note that the electric field $\vec{E}$ in Eq. (1.45) is the *total* electric field, i.e., the electric field that one would measure inside the medium. In other words, the electric field $\vec{E}$ in Eq. (1.30), which we recall as a member of the relation $|\vec{P}| = (\epsilon_r - 1)\, \epsilon_0 \, |\vec{E}|$, constitutes the *total* electric field inside the medium, which needs to vanish for perfect polarization. Modulus-wise, one has $P = (\epsilon_r - 1)\, \epsilon_0 \, (E_{\text{ext}} + E_{\text{ind}})$. That means that $\vec{P}$ must retain a finite value for vanishing $\vec{E}$, which is possible only if $\epsilon_r \to \infty$.

1.2.3 Electric and Magnetic Field Energies

Let us recall basic principles related to field energy densities. To this end, we first recall that if the current density $\vec{J}(\vec{r},t)$ is to be nonvanishing, charges have to move. Then, let us assume that

$$\vec{J}(\vec{r},t) = \frac{\Delta q\, \widehat{v}}{\Delta t\, \Delta A}\,, \tag{1.46}$$

where Δq is the charge moving in the time interval Δt through the cross-sectional area ΔA, in a direction given by the unit velocity vector $\widehat{v}$. Then,

$$\begin{aligned}\vec{E}(\vec{r},t)\cdot\vec{J}(\vec{r},t) &\approx \frac{\Delta q\, \vec{E}(\vec{r},t)\,\cdot\widehat{v}}{\Delta t\,\Delta A} = \frac{(\Delta q\,\vec{E})\cdot\Delta\vec{x}}{\Delta t\,\Delta V} \\ &= \frac{(\Delta\vec{F}_{\rm elec})\cdot\Delta\vec{x}}{\Delta t\,\Delta V} = \frac{\Delta W}{\Delta t\,\Delta V}\end{aligned} \tag{1.47}$$

is the mechanical work done on the moving charges per unit time and unit volume, by the electric force acting on the charges due to the electric field. This quantity has the same physical dimension as the time derivative of an energy density. Let us therefore identify

$$\vec{E}(\vec{r},t)\cdot\vec{J}(\vec{r},t) = \frac{\partial P(\vec{r},t)}{\partial V} \equiv p(\vec{r},t) \tag{1.48}$$

with the local power density $p(\vec{r},t)$ corresponding to the work done on the moving charges by the field. From our experience with mechanical systems, we deduce that the power density supplied to the charged particles of a charge current $\vec{J}_0(\vec{r},t)$ in an electric field $\vec{E}(\vec{r},t)$ is $p(\vec{r},t) = \vec{E}(\vec{r},t)\cdot\vec{J}_0(\vec{r},t)$. We use the result (exercise!)

$$\vec{\nabla}\cdot\left[\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)\right] = -\vec{E}(\vec{r},t)\cdot\vec{\nabla}\times\vec{H}(\vec{r},t) + \vec{H}(\vec{r},t)\cdot\vec{\nabla}\times\vec{E}(\vec{r},t) \tag{1.49}$$

which can be shown by reference to the Levi-Civita tensor ϵ_{ijk}, with

$$\epsilon_{123} = 1\,. \tag{1.50}$$

From the Maxwell equations and using the following identity,

$$\begin{aligned}\vec{\nabla}\cdot\left[\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)\right] &= -\,\vec{E}(\vec{r},t)\cdot\vec{\nabla}\times\vec{H}(\vec{r},t) + \vec{H}(\vec{r},t)\cdot\vec{\nabla}\times\vec{E}(\vec{r},t) \\ &= -\,\vec{E}(\vec{r},t)\cdot\left(\frac{\partial}{\partial t}\vec{D}(\vec{r},t) + \vec{J}_0(\vec{r},t)\right) - \vec{H}(\vec{r},t)\cdot\frac{\partial}{\partial t}\vec{B}(\vec{r},t)\;,\end{aligned} \tag{1.51}$$

we obtain,

$$\vec{\nabla}\cdot\left[\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)\right] + \left[\vec{H}(\vec{r},t)\cdot\frac{\partial}{\partial t}\vec{B}(\vec{r},t) + \vec{E}(\vec{r},t)\cdot\frac{\partial}{\partial t}\vec{D}(\vec{r},t)\right] + \vec{E}(\vec{r},t)\cdot\vec{J}_0(\vec{r},t) = 0\,. \tag{1.52}$$

This has the form of a conservation law,

$$\boxed{\text{Poynting's Theorem:}}\qquad \vec{\nabla}\cdot\vec{S}(\vec{r},t) + \frac{\partial}{\partial t}\,u(\vec{r},t) + \vec{E}(\vec{r},t)\cdot\vec{J}_0(\vec{r},t) = 0\,, \tag{1.53}$$

and is, in some sense, a phenomenological version of Poynting's theorem, where the mechanical power density $\vec{E}(\vec{r},t)\cdot\vec{J}_0(\vec{r},t)$ is exclusively concerned with the free current density $\vec{J}_0(\vec{r},t)$. A phenomenological interpretation is this: Imagine the integrated version of this theorem, over a small sample volume δV. The divergence of the Poynting vector $\vec{S}$ measures the rate at which energy density leaves the sample volume. If this divergence is positive, then field energy leaves the sample volume. This is either compensated by a loss in the energy density, with a negative time derivative of u, or by a negative power density, where mechanical work is done on the moving charges against the direction of the electric field, i.e., $\vec{E}(\vec{r},t) \cdot \vec{J}_0(\vec{r},t) < 0$. (Note that $\vec{E}(\vec{r},t) \cdot \vec{J}_0(\vec{r},t)$ measures the power density for work performed on the moving charges by the field; this work otherwise leads to an increase in the kinetic energy of the charges inside the reference volume.)

The energy current density (energy/unit area/unit time) can be identified as the Poynting vector

$$\vec{S}\left(\vec{r},t\right) = \vec{E}\left(\vec{r},t\right) \times \vec{H}\left(\vec{r},t\right) \,, \tag{1.54}$$

and the time rate of change of the energy density stored in the fields and the medium in the region of the fields is

$$\frac{\partial}{\partial t} u\left(\vec{r},t\right) = \vec{E}\left(\vec{r},t\right) \cdot \frac{\partial}{\partial t}\vec{D}\left(\vec{r},t\right) + \vec{H}\left(\vec{r},t\right) \cdot \frac{\partial}{\partial t}\vec{B}\left(\vec{r},t\right) \,. \tag{1.55}$$

It is interesting to examine in detail the rate of change of the energy density. We first consider the energy density associated with the electric field, using $\vec{D}\left(\vec{r},t\right) = \epsilon_0 \vec{E}\left(\vec{r},t\right) + \vec{P}\left(\vec{r},t\right)$,

$$\begin{aligned}
\frac{\partial}{\partial t} u_E\left(\vec{r},t\right) &= \vec{E}\left(\vec{r},t\right) \cdot \frac{\partial}{\partial t}\vec{D}\left(\vec{r},t\right) && (1.56)\\
&= \vec{E}\left(\vec{r},t\right) \cdot \frac{\partial}{\partial t}\left(\epsilon_0 \vec{E}\left(\vec{r},t\right) + \vec{P}\left(\vec{r},t\right)\right) && (1.57)\\
&= \epsilon_0 \vec{E}\left(\vec{r},t\right) \cdot \frac{\partial}{\partial t}\vec{E}\left(\vec{r},t\right) + \vec{E}\left(\vec{r},t\right) \cdot \vec{J}_p\left(\vec{r},t\right) && (1.58)\\
&= \frac{\epsilon_0}{2}\frac{\partial}{\partial t}\vec{E}^2\left(\vec{r},t\right) + \vec{E}\left(\vec{r},t\right) \cdot \vec{J}_p\left(\vec{r},t\right) \,. && (1.59)
\end{aligned}$$

The second term on the right-hand side then is the power going into the "mechanical" energy of the system. The relationship between the electric field and the polarization is generally very interesting and complicated. In the linear response approximation, the ith Cartesian component of the polarization field and the jth components of the electric field are given by a convolution integral [see Eq. (1.38)],

$$P_i\left(\vec{r},t\right) = \sum_{j=1}^{3} \int \chi_{ij}\left(\vec{r}-\vec{r}\,',t-t'\right)\, E_j\left(\vec{r}\,',t'\right)\, \mathrm{d}^3 r'\, \mathrm{d}t' \,, \tag{1.60}$$

where $\chi_{ij}(\vec{r},t)$ are the components of the electric susceptibility tensor for the system. The energy density time derivative associated with the magnetic field is

$$\frac{\partial}{\partial t} u_M\left(\vec{r},t\right) = \vec{H}\left(\vec{r},t\right) \cdot \frac{\partial}{\partial t}\vec{B}\left(\vec{r},t\right) \,. \tag{1.61}$$

Note that the magnetic force "does no work" (in the Lorentz force, $\vec{B}$ is perpendicular to $\mathrm{d}\vec{r}/\mathrm{d}t$) and, second, the time derivative operates on the source of the force, the magnetic induction vector $\vec{B}$). Using Faraday's law and the relationship $\vec{H}\left(\vec{r},t\right) = \mu_0^{-1}\, \vec{B}\left(\vec{r},t\right) - \vec{M}\left(\vec{r},t\right)$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} u_M\left(\vec{r},t\right) &= \frac{1}{\mu_0}\vec{B}\left(\vec{r},t\right) \cdot \frac{\partial}{\partial t}\vec{B}\left(\vec{r},t\right) - \vec{M}\left(\vec{r},t\right) \cdot \frac{\partial}{\partial t}\vec{B}\left(\vec{r},t\right)\\
&= \frac{1}{\mu_0}\vec{B}\left(\vec{r},t\right) \cdot \frac{\partial}{\partial t}\vec{B}\left(\vec{r},t\right) + \vec{M}\left(\vec{r},t\right) \cdot \vec{\nabla} \times \vec{E}\left(\vec{r},t\right)\\
&= \frac{1}{\mu_0}\vec{B}\left(\vec{r},t\right) \cdot \frac{\partial}{\partial t}\vec{B}\left(\vec{r},t\right) + \vec{\nabla} \cdot \left[\vec{E}\left(\vec{r},t\right) \times \vec{M}\left(\vec{r},t\right)\right] + \vec{E}\left(\vec{r},t\right) \cdot \vec{\nabla} \times \vec{M}\left(\vec{r},t\right)\\
&= \frac{1}{2\mu_0}\frac{\partial}{\partial t}\vec{B}^2\left(\vec{r},t\right) + \vec{\nabla} \cdot \left[\vec{E}\left(\vec{r},t\right) \times \vec{M}\left(\vec{r},t\right)\right] + \vec{E}\left(\vec{r},t\right) \cdot \left[\vec{J}_{\mathrm{mat}} - \vec{J}_p\right] \,. && (1.62)
\end{aligned}$$

The first term on the right-hand side is the power transfer to the magnetic induction field, the second term is the divergence of an energy flux associated with the magnetization of the material, and the last term is the energy supplied to the rotational current density by the electric field generated by the time varying magnetic induction vector.

In terms of the total charge current density, externally controlled (measured) and material response, the time rate of change of the total energy density is

$$\begin{aligned}
\frac{\partial}{\partial t}u\left(\vec{r},t\right) &= \frac{\partial}{\partial t}\left(u_E\left(\vec{r},t\right)+u_M\left(\vec{r},t\right)\right) \\
&= \frac{\epsilon_0}{2}\frac{\partial}{\partial t}\vec{E}^2\left(\vec{r},t\right)+\vec{E}\left(\vec{r},t\right)\cdot\vec{J}_p\left(\vec{r},t\right)+\frac{1}{2\mu_0}\frac{\partial}{\partial t}\vec{B}^2\left(\vec{r},t\right)+\vec{\nabla}\cdot\left[\vec{E}\left(\vec{r},t\right)\times\vec{M}\left(\vec{r},t\right)\right] \\
&\qquad +\vec{E}\left(\vec{r},t\right)\cdot\left[\vec{J}_{\mathrm{mat}}-\vec{J}_p\right] \\
&= \frac{\epsilon_0}{2}\frac{\partial}{\partial t}\left[c^2\,\vec{B}^2\left(\vec{r},t\right)+\vec{E}^2\left(\vec{r},t\right)\right]+\vec{E}\left(\vec{r},t\right)\cdot\vec{J}_{\mathrm{mat}}(\vec{r},t)+\vec{\nabla}\cdot\left[\vec{E}\left(\vec{r},t\right)\times\vec{M}\left(\vec{r},t\right)\right]. \qquad (1.63)
\end{aligned}$$

We recall that

$$\vec{\nabla}\cdot\vec{S}\left(\vec{r},t\right)+\frac{\partial}{\partial t}\,u\left(\vec{r},t\right)+\vec{E}\left(\vec{r},t\right)\cdot\vec{J}_0\left(\vec{r},t\right)=0\,, \qquad (1.64)$$

with the Poynting vector

$$\vec{S}\left(\vec{r},t\right)=\left[\vec{E}\left(\vec{r},t\right)\times\vec{H}\left(\vec{r},t\right)\right]\,. \qquad (1.65)$$

Using the result for $\partial_t u$ in the energy conservation theorem (1.64), we obtain

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E}(\vec{r},t)\times\left[\vec{H}(\vec{r},t)+\vec{M}(\vec{r},t)\right]+\frac{\epsilon_0}{2}\frac{\partial}{\partial t}\left[c^2\,\vec{B}^2\left(\vec{r},t\right)+\vec{E}^2\left(\vec{r},t\right)\right] \\
+\vec{E}\left(\vec{r},t\right)\cdot\underbrace{\left\{\vec{J}_0\left(\vec{r},t\right)+\vec{J}_{\mathrm{mat}}\left(\vec{r},t\right)\right\}}_{=\vec{J}_{\mathrm{total}}(\vec{r},t)}=0\,, \qquad (1.66)
\end{aligned}$$

which amounts to

$$\frac{1}{\mu_0}\vec{\nabla}\cdot\left[\vec{E}\left(\vec{r},t\right)\times\vec{B}\left(\vec{r},t\right)\right]+\frac{\epsilon_0}{2}\frac{\partial}{\partial t}\left(c^2\,\vec{B}^2\left(\vec{r},t\right)+\vec{E}^2\left(\vec{r},t\right)\right)+\vec{E}\left(\vec{r},t\right)\cdot\vec{J}_{\mathrm{total}}\left(\vec{r},t\right)=0\,. \qquad (1.67)$$

This is the microscopic equation for the conservation of energy known as Poynting's theorem, where the mechanical power density is the full term $\vec{E}\left(\vec{r},t\right)\cdot\vec{J}_{\mathrm{total}}\left(\vec{r},t\right)=0$.

The transfer of energy between the fields and the mechanical system is carried out by the interaction of the electric field and the charge current density. The energy transferred to the current can reside in the translational kinetic energy of the system, the potential energy of interaction of the particles of the system, or thermal energy. We will return to this equation as we consider models of systems.

1.2.4 Vector and Scalar Potentials

In the following, we will restrict our investigation to the microscopic form of Maxwell's equations. We start with the two source free equations. First, we consider Gauss's law for the magnetic induction, $\vec{\nabla}\cdot\vec{B}=0$. In this case, a vector potential, $\vec{A}\left(\vec{r},t\right)$, can be defined such that the magnetic induction vector is given by

Magnetic Field and Vector Potential: $$\vec{B}\left(\vec{r},t\right)=\vec{\nabla}\times\vec{A}\left(\vec{r},t\right)\,. \qquad (1.68)$$

The vector potential is then used in Faraday's law

$$\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}\vec{B} = 0 \tag{1.69}$$

to yield

$$\vec{\nabla} \times \vec{E}\,(\vec{r},t) + \frac{\partial}{\partial t}\left[\vec{\nabla} \times \vec{A}\,(\vec{r},t)\right] = \vec{\nabla} \times \left[\vec{E}\,(\vec{r},t) + \frac{\partial}{\partial t}\vec{A}\,(\vec{r},t)\right] = \vec{0}\,. \tag{1.70}$$

It follows that the curl of the vector field $\vec{E}\,(\vec{r},t) + \frac{\partial}{\partial t}\vec{A}\,(\vec{r},t)$ vanishes, which is why we can find a scalar potential $\Phi\,(\vec{r},t)$ such that

$$\vec{E}\,(\vec{r},t) + \frac{\partial}{\partial t}\vec{A}\,(\vec{r},t) = -\vec{\nabla}\Phi\,(\vec{r},t) \tag{1.71}$$

and

Electric Field and Scalar/Vector Potential: $$\vec{E}\,(\vec{r},t) = -\vec{\nabla}\Phi\,(\vec{r},t) - \frac{\partial}{\partial t}\vec{A}\,(\vec{r},t)\,. \tag{1.72}$$

The fields constructed according to Eqs. (1.68) and (1.72) automatically fulfill the homogeneous Maxwell equations $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = \vec{0}$.

Remark. Relativistically, the four space-time components $x^\mu = (c\,t, \vec{r})$ constitute a so-called four-vector with defined transformation properties under Lorentz transformations. Also, $A^\mu = (\Phi, c\,\vec{A})$ constitutes a four-vector. The quantities $\vec{E}$ and $c\,\vec{B}$ are also related. We can then write

$$\vec{E}\,(\vec{r},t) = -\vec{\nabla}\Phi\,(\vec{r},t) - \frac{\partial}{\partial(c\,t)}\left[c\,\vec{A}\,(\vec{r},t)\right] \tag{1.73}$$

in order to convince ourselves of the consistency of the units used.

Gauss's law for the electric field and the Ampere-Maxwell law now provide the equations which the vector and scalar potential must satisfy, i.e., they couple them to their sources, which are the charge and current densities. First we consider Gauss' law and obtain

$$\vec{\nabla} \cdot \vec{E} = -\vec{\nabla}^2\Phi\,(\vec{r},t) - \frac{\partial}{\partial t}\vec{\nabla} \cdot \vec{A}\,(\vec{r},t) = \frac{1}{\epsilon_0}\rho\,(\vec{r},t)\,, \tag{1.74}$$

while the Ampere-Maxwell law

$$\vec{\nabla} \times \vec{B}\,(\vec{r},t) = \vec{\nabla} \times \left[\vec{\nabla} \times \vec{A}\,(\vec{r},t)\right] = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}\,(\vec{r},t) + \mu_0\,\vec{J}\,(\vec{r},t) \tag{1.75}$$

becomes

$$\begin{aligned}
\vec{\nabla} \times \left[\vec{\nabla} \times \vec{A}\,(\vec{r},t)\right] + \frac{1}{c^2}\,\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\vec{A}\,(\vec{r},t) + \vec{\nabla}\Phi\,(\vec{r},t)\right] &= \mu_0\,\vec{J}\,(\vec{r},t) \\
\Leftrightarrow\ \vec{\nabla}\left\{\vec{\nabla} \cdot \vec{A}\,(\vec{r},t)\right\} - \vec{\nabla}^2\vec{A}\,(\vec{r},t) + \frac{1}{c^2}\,\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\vec{A}\,(\vec{r},t) + \vec{\nabla}\Phi\,(\vec{r},t)\right] &= \mu_0\,\vec{J}\,(\vec{r},t) \\
\Leftrightarrow\ \vec{\nabla}\left\{\vec{\nabla} \cdot \vec{A}\,(\vec{r},t) + \frac{1}{c^2}\,\frac{\partial}{\partial t}\Phi\,(\vec{r},t)\right\} - \vec{\nabla}^2\vec{A}\,(\vec{r},t) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}\,(\vec{r},t) &= \mu_0\,\vec{J}\,(\vec{r},t)\,.
\end{aligned} \tag{1.76}$$

We here assume that $\vec{J}$ is the total current density, i.e., the current density we would use in the fundamental theory, where we do not use the phenomenological Maxwell equations involving the $\vec{D}$ and $\vec{H}$ fields. For Eqs. (1.74) and (1.76), we can conclude that the scalar and vector potentials are coupled to the sources as follows,

Coupling to Sources: $$-\vec{\nabla}^2\Phi\,(\vec{r},t) - \frac{\partial}{\partial t}\vec{\nabla} \cdot \vec{A}\,(\vec{r},t) = \frac{1}{\epsilon_0}\rho\,(\vec{r},t)\,, \tag{1.77a}$$

$$\vec{\nabla}\left\{\vec{\nabla} \cdot \vec{A}\,(\vec{r},t) + \frac{1}{c^2}\,\frac{\partial}{\partial t}\Phi\,(\vec{r},t)\right\} - \vec{\nabla}^2\vec{A}\,(\vec{r},t) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}\,(\vec{r},t) = \mu_0\,\vec{J}\,(\vec{r},t)\,. \tag{1.77b}$$

These are coupled second order partial differential equations.

1.2.5 Gauge Invariance and Lorenz Gauge

If one measures lengths or heights, one always has the freedom to choose the zero point of measurement. If one is to hang a picture on a wall, a convenient zero point for a height measurement is the floor of the room, while, for measuring the altitude of a mountain, a convenient reference point is the sea level. A trivial gauge transformation which leaves the electric field invariant is simply $\Phi \to \Phi + \mathrm{const.}$, where $\mathrm{const.}$ is a constant potential, which does not affect the gradient ("global" gauge transformation). This amounts to choosing a zero for the measurement of the "height of" the potential. Now, we are in principle free to choose the zeros for the measurements of the potentials any way we want, and moreover, to choose these zero points differently at every space-time point. This is called a "local" gauge transformation. Let us now explore this concept in detail.

Since the fields only depend on derivatives of the potentials, we have the freedom to choose the potentials so as to provide a convenient form for the equations for the potentials. The addition of a term $\vec{A}(\vec{r},t) \to \vec{A}(\vec{r},t) + \vec{\nabla}\Lambda(\vec{r},t)$ to the vector potential does not change the magnetic induction field, because

$$\vec{\nabla} \times \vec{\nabla}\Lambda(\vec{r},t) = \vec{0}. \tag{1.78}$$

In this case, we obtain the new vector potential (denoted by a prime)

$$\boxed{\text{Gauge Transform of Vector Potential:}} \quad \vec{A}'(\vec{r},t) = \vec{A}(\vec{r},t) + \vec{\nabla}\Lambda(\vec{r},t) \,. \tag{1.79}$$

The magnetic field is invariant under the gauge transform of the vector potential. We now insert the equation $\vec{A}(\vec{r},t) = \vec{A}'(\vec{r},t) - \vec{\nabla}\Lambda(\vec{r},t)$ for the gauge transformed vector potential into the formula Eq. (1.71) for the electric field,

$$\begin{aligned}\vec{E}(\vec{r},t) &= -\frac{\partial}{\partial t}\vec{A}'(\vec{r},t) + \frac{\partial}{\partial t}\vec{\nabla}\Lambda(\vec{r},t) - \vec{\nabla}\Phi(\vec{r},t) \\ &= -\frac{\partial}{\partial t}\vec{A}'(\vec{r},t) - \vec{\nabla}\left\{-\frac{\partial}{\partial t}\Lambda(\vec{r},t) + \Phi(\vec{r},t)\right\} .\end{aligned} \tag{1.80}$$

Changing the vector potential will change the electric field unless the scalar potential is also changed to the term that appears in curly brackets,

$$\boxed{\text{Gauge Transform of Scalar Potential:}} \qquad \Phi'(\vec{r},t) = \Phi(\vec{r},t) - \frac{\partial}{\partial t}\Lambda(\vec{r},t) \,, \tag{1.81}$$

so that

$$\begin{aligned}\vec{E}(\vec{r},t) &= -\frac{\partial}{\partial t}\vec{A}'(\vec{r},t) - \vec{\nabla}\Phi'(\vec{r},t) \\ &= -\frac{\partial}{\partial t}\left(\vec{A}(\vec{r},t) + \vec{\nabla}\Lambda(\vec{r},t)\right) - \vec{\nabla}\left[\Phi(\vec{r},t) - \partial_t\Lambda(\vec{r},t)\right] \\ &= -\frac{\partial}{\partial t}\vec{A}(\vec{r},t) - \vec{\nabla}\Phi(\vec{r},t) \,.\end{aligned} \tag{1.82}$$

(The field is gauge invariant, the potential is not.) The equations (1.79) and (1.81),

$$\vec{A}'(\vec{r},t) = \vec{A}(\vec{r},t) + \vec{\nabla}\Lambda(\vec{r},t) \,, \qquad \Phi'(\vec{r},t) = \Phi(\vec{r},t) - \frac{\partial}{\partial t}\Lambda(\vec{r},t) \,, \tag{1.83}$$

define the gauge transformation of the potentials. The electric and magnetic fields, which are invariant under the gauge transformations, are said to be gauge invariant. One assumes that they should be gauge invariant as they constitute physically observable field strengths.

One possible requirement on the new potentials is the

$$\boxed{\textbf{Lorenz Gauge Condition:}} \qquad \vec{\nabla}\cdot\vec{A}'(\vec{r},t)+\frac{1}{c^2}\frac{\partial}{\partial t}\Phi'(\vec{r},t)=0\,, \tag{1.84}$$

which is commonly known as the Lorenz gauge condition. How can Λ be constructed to make the gauge transformed vector and scalar potentials fulfill the gauge condition? Well, given potentials $\vec{A}$ and Φ which do not satisfy the Lorenz gauge condition, we have

$$\vec{\nabla}\cdot\left[\vec{A}(\vec{r},t)+\vec{\nabla}\Lambda(\vec{r},t)\right]+\frac{1}{c^2}\frac{\partial}{\partial t}\left[\Phi(\vec{r},t)-\frac{\partial}{\partial t}\Lambda(\vec{r},t)\right]=0\,, \tag{1.85}$$

or

$$\vec{\nabla}\cdot\vec{A}(\vec{r},t)+\vec{\nabla}^2\Lambda(\vec{r},t)+\frac{1}{c^2}\frac{\partial}{\partial t}\Phi(\vec{r},t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Lambda(\vec{r},t)=0\,, \tag{1.86}$$

and when we rearrange the terms with Λ to lie on the left-hand side, we have

$$\boxed{\textbf{Gauge Transform Function:}} \quad \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\vec{\nabla}^2\right)\Lambda(\vec{r},t)=\vec{\nabla}\cdot\vec{A}(\vec{r},t)+\frac{1}{c^2}\frac{\partial}{\partial t}\Phi(\vec{r},t)\ , \tag{1.87}$$

i.e., Λ satisfies the wave equation with the source determined by the potentials we wish to modify. We can now reverse the argument as follows: Because it is always possible to find an adequate solution to Eq. (1.87), we do not throw away important solutions to the Maxwell equations if we impose the Lorentz gauge condition in the first place, i.e., before we seek to solve the inhomogeneous Maxwell equations. This is, in some sense, like trying to put the saddle onto the horse from the back. The horse is benign, so it works.

If the vector and scalar potentials, $\vec{A}$ and Φ, were to fulfill the Lorenz condition in the first place, then the right-hand side of Eq. (1.87) would vanish. The gauge transform function Λ has to fulfill an inhomogeneous wave equation in which the source term is just equivalent to the left-hand side of the gauge condition. That means that if, accidentally, $\vec{A}$ and Φ were to fulfill the Lorentz condition even before being gauge transformed, then we could choose for Λ any function that fulfills the homogeneous wave equation, and still retain fields that fulfill the Lorentz condition, after the gauge transformation.

We recall that according to Eqs. (1.74) and (1.76), Gauss's law and the Ampere-Maxwell law relate the sources to the fields. In terms of the primed, gauge-transformed quantities, we have

$$-\vec{\nabla}^2\Phi'(\vec{r},t)-\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}'(\vec{r},t)=\frac{1}{\epsilon_0}\rho(\vec{r},t)\ , \tag{1.88}$$

$$\vec{\nabla}\underbrace{\left\{\vec{\nabla}\cdot\vec{A}'(\vec{r},t)+\frac{1}{c^2}\frac{\partial}{\partial t}\Phi'(\vec{r},t)\right\}}_{=0}-\vec{\nabla}^2\vec{A}'(\vec{r},t)+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}'(\vec{r},t)=\mu_0\,\vec{J}(\vec{r},t)\ . \tag{1.89}$$

If the primed potentials fulfill the Lorenz gauge condition (1.84),

$$\vec{\nabla}\cdot\vec{A}'(\vec{r},t)+\frac{1}{c^2}\frac{\partial}{\partial t}\Phi'(\vec{r},t)=0\,, \tag{1.90}$$

the equations are uncoupled. For the second equation, this is immediately obvious, and for the first equation, we can see this according to

$$-\vec{\nabla}^2\Phi'(\vec{r},t)-\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}'(\vec{r},t)=-\vec{\nabla}^2\Phi'(\vec{r},t)+\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\Phi\right)=\frac{1}{\epsilon_0}\rho(\vec{r},t)\ . \tag{1.91}$$

The equations thus take the form of uncoupled, inhomogeneous wave equations,

$$\boxed{\textbf{Lorenz Gauge/Scalar Potential:}} \qquad \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right) \Phi'(\vec{r},t) = \frac{1}{\epsilon_0}\,\rho(\vec{r},t)\,, \tag{1.92a}$$

$$\boxed{\textbf{Lorenz Gauge/Vector Potential:}} \qquad \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right) \vec{A}'(\vec{r},t) = \mu_0\,\vec{J}(\vec{r},t)\,. \tag{1.92b}$$

Once these are solved, we can compute the electric and magnetic fields according to Eqs. (1.72) and (1.68). Both the scalar and vector potentials satisfy a wave equation with the respective sources the charge density and the charge current density. In a source-free region of space-time, the scalar and vector potentials thus satisfy homogeneous wave equations. This finding suggests the existence of electromagnetic waves.

Remark. In the fully relativistic formalism, we may define the "quabla" operator as

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\,. \tag{1.93}$$

Furthermore, the 4-vector potential is $A^\mu = (\Phi, c\,\vec{A})$, which summarizes the scalar potential and the vector potential into a Lorentz-covariant 4-vector. This means that if we have a Lorentz transformation that transforms the space-time coordinates $x^\mu = (c\,t, \vec{r})$ according to

$$x'^\mu = L^\mu_\nu\, x^\nu \tag{1.94}$$

then the 4-vector potentials, as seen from the moving observer, needs to be transformed in the same way,

$$A'^\mu = L^\mu_\nu\, A^\nu\,. \tag{1.95}$$

Note that here, the primed quantities refer to those seen in a different Lorentz frame (not the gauge transformed quantities). Then, in units with $\hbar = c = \epsilon_0 = 1$ (Heaviside–Lorentz units, or just "natural units"), the two equations (1.92a) and (1.92b) can be summarized into one,

$$\Box A^\mu = J^\mu\,, \qquad J^\mu = \left(c\,\rho, \vec{J}\right)\,. \tag{1.96}$$

1.2.6 "Gauge Always Shoots Twice"

Let us recall a few formulas related to gauge transformations. Equation (1.84) is called the Lorentz condition,

$$\vec{\nabla}\cdot\vec{A}'(\vec{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi'(\vec{r},t) = 0\,. \tag{1.97}$$

The gauge transformed fields,

$$\vec{A}'(\vec{r},t) = \vec{A}(\vec{r},t) + \vec{\nabla}\Lambda(\vec{r},t)\,, \qquad \Phi'(\vec{r},t) = \Phi(\vec{r},t) - \frac{\partial}{\partial t}\Lambda(\vec{r},t)\,, \tag{1.98}$$

fulfill this condition provided the Λ function fulfills an inhomogeneous wave equation,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\Lambda(\vec{r},t) = \vec{\nabla}\cdot\vec{A}(\vec{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi(\vec{r},t)\,. \tag{1.99}$$

If $\vec{A}$ and Φ accidentally fulfill the Lorentz condition, already, then it is possible to do a second gauge transform, using a function Λ, and to still remain within the family of potentials that fulfill the Lorenz gauge condition.

The Lorenz gauge condition is a first-order partial differential equation imposed on the 4-vector potential. It is not an algebraic condition. We know that a partial differential equation allows for many solutions. Therefore, Eq. (1.84) does not determine the 4-vector potential uniquely, and even given the Lorentz condition, we still have a certain freedom to choose our 4-vector potential within the "Lorenz gauge family" without affecting the electric and magnetic fields. This is sometimes expressed by saying that "gauge always shoots twice."

Therefore, if $\vec{A}'$ and Φ' satisfy the Lorentz condition (1.84) and Λ' is yet another solution of the homogeneous wave equation

$$\boxed{\textbf{Homogeneous Wave Equation:}} \qquad \left(\frac{1}{c}\,\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\Lambda'\,(\vec{r},t) = 0\,, \tag{1.100}$$

then

$$\boxed{\textbf{Second Gauge Transform of Vector Potential:}} \quad \vec{A}''\,(\vec{r},t) = \vec{A}'\,(\vec{r},t) + \vec{\nabla}\Lambda'\,(\vec{r},t)\;, \tag{1.101}$$

$$\boxed{\textbf{Second Gauge Transform of Scalar Potential:}} \quad \Phi''\,(\vec{r},t) = \Phi'\,(\vec{r},t) - \frac{\partial}{\partial t}\Lambda'\,(\vec{r},t)\;. \tag{1.102}$$

The second gauge transformation defines new vector and scalar potentials which still satisfy the Lorentz condition and give the same $\vec{E}$ and $\vec{B}$ fields. The class of all such combinations of vector and scalar potentials related by this restricted gauge transformation belongs to the Lorenz gauge.

Choosing potentials which satisfy the Lorentz condition not only uncouples the equations for the potentials but also yields equations which are invariant under the Lorentz/Poincaré transformations, as is evident from Eq. (1.96). Both sides of the equation (1.96) carry only one Lorentz index.

1.2.7 Coulomb Gauge

Another gauge which is often used in radiation theory is the Coulomb, transverse, or radiation gauge. The gauge condition is

$$\boxed{\textbf{Coulomb/Radiation Gauge (Version 1):}} \qquad \vec{\nabla}\cdot\vec{A}\,(\vec{r},t) = 0\,. \tag{1.103a}$$

A well-known theorem from vector analysis says that any vector field $\vec{A} = \vec{A}\,(\vec{r},t)$ can be uniquely decomposed into a transverse component $\vec{A}_\perp\,(\vec{r},t)$ whose divergence vanishes, $\vec{\nabla}\cdot\vec{A}_\perp\,(\vec{r},t) = 0$, and a longitudinal component $\vec{A}_\parallel\,(\vec{r},t)$, whose curl vanishes, $\vec{\nabla}\cdot\vec{A}_\perp\,(\vec{r},t) = 0$. The decomposition is discussed below in Sec. 1.2.8. As a consequence, one may express the radiation gauge condition in two alternative forms,

$$\boxed{\textbf{Coulomb/Radiation Gauge (Version 2):}} \qquad \vec{A}\,(\vec{r},t) = \vec{A}_\perp\,(\vec{r},t)\;, \tag{1.103b}$$

$$\boxed{\textbf{Coulomb/Radiation Gauge (Version 3):}} \qquad \vec{A}_\parallel\,(\vec{r},t) = 0\,. \tag{1.103c}$$

The three versions (1.103a), (1.103b) and (1.103c) are equivalent. Version (1.103a) corresponds to the original Coulomb gauge, whereas version (1.103b) suggests the name "transverse gauge", because the vector potential is transverse.

In general, the vector and scalar potentials are coupled to the sources as follows,

$$-\vec{\nabla}^2\Phi\,(\vec{r},t) - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}\,(\vec{r},t) = \frac{1}{\epsilon_0}\rho\,(\vec{r},t)\;, \tag{1.104a}$$

$$\vec{\nabla}\left\{\vec{\nabla}\cdot\vec{A}\,(\vec{r},t) + \frac{1}{c^2}\,\frac{\partial}{\partial t}\Phi\,(\vec{r},t)\right\} - \vec{\nabla}^2\vec{A}\,(\vec{r},t) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}\,(\vec{r},t) = \mu_0\,\vec{J}\,(\vec{r},t)\;. \tag{1.104b}$$

In the radiation gauge, the potentials thus satisfy the equations

Radiation Gauge/Scalar Potential:

$$-\vec{\nabla}^2\Phi(\vec{r},t) = \frac{1}{\epsilon_0}\rho(\vec{r},t) \,, \tag{1.105a}$$

Radiation Gauge/Vector Potential:

$$-\vec{\nabla}^2\vec{A}(\vec{r},t) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}(\vec{r},t) = \mu_0\,\vec{J}(\vec{r},t) - \frac{1}{c^2}\vec{\nabla}\left[\frac{\partial}{\partial t}\Phi(\vec{r},t)\right] . \tag{1.105b}$$

We reemphasize that these equations are only valid in the radiation gauge (1.103). The equation for the vector potential can be simplified further and written in terms of two, simpler equations. Let us write Eq. (1.105b) as follows,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\vec{A}(\vec{r},t) = \mu_0\,\vec{J}(\vec{r},t) - \frac{1}{c^2}\vec{\nabla}\partial_t\Phi(\vec{r},t) \,, \tag{1.106}$$

before we separate this relation into transverse and lognitudinal components. Indeed, the transverse and longitudinal components of the expression

$$\vec{F}(\vec{r},t) = \frac{1}{c^2}\vec{\nabla}\partial_t\Phi(\vec{r},t) = \vec{\nabla}\left(\frac{1}{c^2}\partial_t\Phi(\vec{r},t)\right) \tag{1.107}$$

now have to be identified. This is easy because $\vec{F}(\vec{r},t)$ is a gradient vector, and so

$$\vec{\nabla}\times\vec{F}(\vec{r},t) = 0\,, \qquad \vec{F}(\vec{r},t) = \vec{F}_{\parallel}(\vec{r},t)\,. \tag{1.108}$$

The transverse component of Eq. (1.106) thus reads as

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\vec{A}_{\perp}(\vec{r},t) = \mu_0\,\vec{J}_{\perp}(\vec{r},t) \,. \tag{1.109}$$

where we have used the following decomposition of the current density into transverse ($\perp$) and longitudinal ($\parallel$) components,

$$\vec{J} = \vec{J}_{\perp} + \vec{J}_{\parallel}\,, \qquad \vec{\nabla}\cdot\vec{J}_{\perp}(\vec{r},t) = 0\,, \qquad \vec{\nabla}\times\vec{J}_{\parallel}(\vec{r},t) = 0\,. \tag{1.110}$$

The longitudinal component of Eq. (1.106) is found as

$$\mu_0\,\vec{J}_{\parallel}(\vec{r},t) = \frac{1}{c^2}\vec{\nabla}\partial_t\Phi(\vec{r},t) \,, \qquad \vec{J}_{\parallel}(\vec{r},t) = \epsilon_0\vec{\nabla}\partial_t\Phi(\vec{r},t) \,, \tag{1.111}$$

where we have used that $\mu_0\,\epsilon_0 = 1/c^2$. The divergence of this relation is easily identified as the charge conservation condition,

$$\vec{\nabla}\cdot\vec{J}_{\parallel}(\vec{r},t) = \vec{\nabla}\cdot\vec{J}(\vec{r},t) = \epsilon_0\partial_t\vec{\nabla}^2\Phi(\vec{r},t) = \epsilon_0\frac{\partial}{\partial t}\left(-\frac{1}{\epsilon_0}\rho(\vec{r},t)\right) = -\frac{\partial}{\partial t}\rho(\vec{r},t) \,. \tag{1.112}$$

The longitudinal current thus fulfills the following equation,

Longitudinal Current:

$$\vec{J}_{\parallel} = \epsilon_0\,\vec{\nabla}\left[\frac{\partial}{\partial t}\Phi(\vec{r},t)\right] . \tag{1.113}$$

We can thus establish that the following three equations couple the scalar and vector potentials to their sources in the Coulomb gauge,

Radiation Gauge/Scalar Potential:

$$-\vec{\nabla}^2\Phi(\vec{r},t) = \frac{1}{\epsilon_0}\rho(\vec{r},t) \,, \tag{1.114a}$$

Radiation Gauge/Vector Potential:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\vec{A}_{\perp}(\vec{r},t) = \mu_0\,\vec{J}_{\perp}(\vec{r},t) \,, \tag{1.114b}$$

Radiation Gauge/Longitudinal Current:

$$\epsilon_0\,\vec{\nabla}\left[\frac{\partial}{\partial t}\Phi(\vec{r},t)\right] = \vec{J}_{\parallel}(\vec{r},t)\,. \tag{1.114c}$$

In particular, the vector potentials only couple to the transverse current.

Let us now consider, just like for the Lorenz gauge, a second gauge transformation within the family of the radiation gauge. So, we now consider the effect of a second gauge transformation within the class of radiation gauges and write Λ' as χ,

$$\boxed{\textbf{Second Gauge Transform of Vector Potential:}} \quad \vec{A}''(\vec{r},t) = \vec{A}'(\vec{r},t) + \vec{\nabla}\chi(\vec{r},t) \,, \tag{1.115a}$$

$$\boxed{\textbf{Second Gauge Transform of Scalar Potential:}} \quad \Phi''(\vec{r},t) = \Phi'(\vec{r},t) - \frac{\partial}{\partial t}\chi(\vec{r},t) \,. \tag{1.115b}$$

From the gauge condition imposed on the second transformed vector potential,

$$\vec{\nabla} \cdot \vec{A}''(\vec{r},t) = 0 \,, \qquad \vec{\nabla}^2 \chi(\vec{r},t) = 0 \,, \tag{1.116}$$

we see that the divergence of χ has to vanish. Conversely, any function $\chi = \chi(\vec{r},t)$ that fulfills $\vec{\nabla}^2\chi(\vec{r},t) = 0$ mediates a second gauge transform of the vec and scalar potentials within the radiation gauge.

Remark. The radiation gauge is well suited for describing the propagation of radiation in the absence of sources, i.e., in a region of space where there are no charges, and no currents. If sources are absent, then $\vec{\nabla}^2\Phi(\vec{r},t) = 0$, and therefore $\Phi(\vec{r},t) = 0$, and so

$$\vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{A}(\vec{r},t) \,, \tag{1.117a}$$

$$\vec{B}(\vec{r},t) = \vec{\nabla} \times \vec{A}(\vec{r},t) \,. \tag{1.117b}$$

Let us consider a typical vector potential describing a travelling electromagnetic wave,

$$\vec{A}(\vec{r},t) = \hat{\mathrm{e}}_{\vec{k}\lambda}\, A_0 \cos(\vec{k}\cdot\vec{r} - \omega t) \,, \tag{1.118}$$

where $\vec{k}$ is the wave vector, and $\lambda = 1,2$ is the polarization. Then,

$$\vec{\nabla} \cdot \vec{A}(\vec{r},t) = -\left(\vec{k}\cdot\hat{\mathrm{e}}_{\vec{k}\lambda}\right) A_0 \sin(\vec{k}\cdot\vec{r} - \omega t) = 0 \,, \tag{1.119}$$

holds because the polarization vector is always perpendicular to the propagation vector, $\vec{k}\cdot\hat{\mathrm{e}}_{\vec{k}\lambda}$.

Remark. However, the gauge is also called the Coulomb gauge. This is because the equation that couples the electrostatic potential to the source,

$$\vec{\nabla}^2\Phi(\vec{r},t) = -\frac{1}{\epsilon_0}\,\rho(\vec{r},t) \,, \tag{1.120}$$

has the instantaneous, action-at-distance solution

$$\Phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3r'\, \frac{1}{|\vec{r}-\vec{r}'|}\,\rho(\vec{r}',t) \,. \tag{1.121}$$

The similarity of this solution with the Coulomb law, i.e., the simple presence of the factor $\frac{1}{|\vec{r}-\vec{r}'|}$, implies the nomenclature of Coulomb gauge. Moreover, in a time-independent situation, the dependence on t cancels, and the Poisson equation from electrostatics is immediately recovered. However, for time-dependent problems, the action at a distance poses important questions. If $\vec{r}'$ is in the Andromeda Nebula and $\vec{r}$ is near Alpha Centauri, then we have a problem with causality because the field is generated at the same point in time as the source acts, namely, at time t. We note that the scalar potential in this gauge responds instantaneously, at each point in space, to any temporal variation in the charge density. This can be compared with the equivalent description furnished by working in the Lorenz gauge. In the Lorenz gauge, we will find that the effects of a temporal variation in the sources propagate through the potentials with a finite speed. [At a later stage in the course, we may reexamine this point, based on the non-local relation of the longitudinal component of the electric field, $E_{\parallel}(\vec{r},t) = -\vec{\nabla}\Phi(\vec{r},t)$, and the full electric field $E(\vec{r},t)$.]

1.2.8 Separation of a Vector Field into Transverse and Longitudinal Components

Let $\vec{J}\,(\vec{r},t)$ be a vector field which we would like to separate into transverse and longitudinal components. We seek a separation

$$\vec{J}\,(\vec{r},t) = \vec{J}_\perp\,(\vec{r},t) + \vec{J}_\parallel\,(\vec{r},t)\,, \qquad \vec{\nabla}\cdot\vec{J}_\perp\,(\vec{r},t) = 0\,, \qquad \vec{\nabla}\times\vec{J}_\parallel\,(\vec{r},t) = 0\,. \tag{1.122}$$

It is not obvious how to do this. However, "there is always tricks." We first write a seemingly innocent vector identity, which reads

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{J}\,(\vec{r},t)\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{J}\,(\vec{r},t)\right) - \vec{\nabla}^2\vec{J}\,(\vec{r},t)\,. \tag{1.123}$$

We now rewrite the equation just derived in a special way,

$$\vec{\nabla}^2\vec{J}\,(\vec{r},t) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{J}\,(\vec{r},t)\right) - \vec{\nabla}\times\left(\vec{\nabla}\times\vec{J}\,(\vec{r},t)\right) \equiv \vec{F}(\vec{r},t)\,, \tag{1.124}$$

and we now *interpret* the expression $\vec{F}(\vec{r},t) = \vec{\nabla}^2\vec{J}\,(\vec{r},t)$ as a source for the field $\vec{J}\,(\vec{r},t)$ generated by the field itself, in a somewhat redundant manner. We now recall the Green function for Laplace's equation,

$$\vec{\nabla}^2 g\,(\vec{r},\vec{r}\,',t) = \delta^{(3)}(\vec{r}-\vec{r}\,')\,, \tag{1.125}$$

$$g\,(\vec{r},\vec{r}\,',t) = -\frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}\,'|}\,. \tag{1.126}$$

Combining Eqs. (1.124) and (1.126) with vanishing boundaries at infinity, we obtain

$$\begin{aligned}
\vec{J}\,(\vec{r},t) &= \int g\,(\vec{r},\vec{r}\,',t)\;\vec{F}(\vec{r}\,',t)\,\mathrm{d}^3r' \\
&= \int g\,(\vec{r},\vec{r}\,',t)\;\vec{\nabla}'^2\vec{J}\,(\vec{r}\,',t)\;\mathrm{d}^3r' \\
&= -\int \frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}\,'|}\left[\vec{\nabla}'\left(\vec{\nabla}'\cdot\vec{J}\,(\vec{r}\,',t)\right) - \vec{\nabla}'\times\left(\vec{\nabla}'\times\vec{J}\,(\vec{r}\,',t)\right)\right]\mathrm{d}^3r' \\
&= -\frac{1}{4\pi}\int \frac{\vec{\nabla}'\left(\vec{\nabla}'\cdot\vec{J}\,(\vec{r}\,',t)\right)}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}^3r' + \frac{1}{4\pi}\int\frac{\vec{\nabla}'\times\left(\vec{\nabla}'\times\vec{J}\,(\vec{r}\,',t)\right)}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}^3r' \\
&= +\frac{1}{4\pi}\int\left(\vec{\nabla}'\cdot\vec{J}\,(\vec{r}\,',t)\right)\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}^3r' - \frac{1}{4\pi}\int\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}\,'|}\times\left(\vec{\nabla}'\times\vec{J}\,(\vec{r}\,',t)\right)\mathrm{d}^3r' \\
&= -\frac{1}{4\pi}\vec{\nabla}\int\frac{\vec{\nabla}'\cdot\vec{J}\,(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}^3r' + \frac{1}{4\pi}\vec{\nabla}\times\int\frac{\vec{\nabla}'\times\vec{J}\,(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}^3r'\,.
\end{aligned} \tag{1.127}$$

The longitudinal and transverse parts of the current density, respectively, are thus identified as

Longitudinal Component:
$$\vec{J}_\parallel\,(\vec{r},t) = -\frac{1}{4\pi}\vec{\nabla}\int\frac{\vec{\nabla}'\cdot\vec{J}\,(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}^3r' \tag{1.128}$$

Transverse Component:
$$\vec{J}_\perp\,(\vec{r},t) = \frac{1}{4\pi}\vec{\nabla}\times\int\frac{\vec{\nabla}'\times\vec{J}\,(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}^3r'\,. \tag{1.129}$$

These are highly non-local functions of the full current density $\vec{J}\,(\vec{r},t)$.

1.3 From Dissipation and Energy Storage to the Complex Formalism

1.3.1 Basics of Fourier Transformations

Because the permittivity and permeability functions are most practically formulated in Fourier space, we now need to investigate, briefly, certain basic aspects of Fourier transformations. We have noted that the response of the system to the applied fields is generally a convolution of the fields and a response function of the system. In a large class of experiments, for which the wavelengths of the electromagnetic fields are large compared to the characteristic lengths of the system, the spatial dependence of the response function is approximated by a quantity proportional to $\delta^{(3)}(\vec{r}-\vec{r}')$. For these systems, it is sufficient to work in frequency space. Maxwell's equations are

$$\vec{\nabla}\cdot\vec{\widetilde{D}}\,(\vec{r},\omega) = \widetilde{\rho}_0\,(\vec{r},\omega)\ , \tag{1.130a}$$

$$\vec{\nabla}\cdot\vec{\widetilde{B}}\,(\vec{r},\omega) = 0\,, \tag{1.130b}$$

$$\vec{\nabla}\times\vec{\widetilde{E}}\,(\vec{r},\omega) = \mathrm{i}\,\omega\ \vec{\widetilde{B}}\,(\vec{r},\omega)\ , \tag{1.130c}$$

$$\vec{\nabla}\times\vec{\widetilde{H}}\,(\vec{r},\omega) = \ -\mathrm{i}\,\omega\ \vec{\widetilde{D}}\,(\vec{r},\omega) + \vec{\widetilde{J}}_0\,(\vec{r},\omega)\ . \tag{1.130d}$$

These equations, for the Fourier transforms marked by a tilde, are obtained from the ordinary Maxwell equations by setting (Fourier transformation with respect to time, not to space)

$$\vec{E}\,(\vec{r},t) = \int\frac{\mathrm{d}\omega}{2\pi}\,\vec{\widetilde{E}}\,(\vec{r},\omega)\ \mathrm{e}^{-\mathrm{i}\omega t}\,, \qquad \vec{E}\,(\vec{r},\omega) = \int\mathrm{d}t\ \vec{\widetilde{E}}\,(\vec{r},t)\ \mathrm{e}^{\mathrm{i}\omega t}\,, \tag{1.131a}$$

$$\vec{B}\,(\vec{r},t) = \int\frac{\mathrm{d}\omega}{2\pi}\,\vec{\widetilde{B}}\,(\vec{r},\omega)\ \mathrm{e}^{-\mathrm{i}\omega t}\,, \qquad \vec{B}\,(\vec{r},\omega) = \int\mathrm{d}t\ \vec{\widetilde{B}}\,(\vec{r},t)\ \mathrm{e}^{\mathrm{i}\omega t}\,, \tag{1.131b}$$

so that the time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\Leftrightarrow -\mathrm{i}\,\omega\,, \qquad \frac{\mathrm{d}}{\mathrm{d}\omega}\Leftrightarrow \mathrm{i}\,t\,. \tag{1.132}$$

A derivative with respect to time is equivalent to a multiplication by $-\mathrm{i}\,\omega$ in Fourier (frequency) space. By contrast, a derivative with respect to frequency is equivalent to a multiplication by $\mathrm{i}\,t$ in Fourier (frequency) space. The Fourier integrals in Eq. (1.131) converge only provided the functions $\vec{\widetilde{E}}\,(\vec{r},t)$ and $\vec{\widetilde{B}}\,(\vec{r},t)$ fall off sufficiently fast for the infinite past and future. This implies that the Fourier-transform based formalism is primarily suited for light pulses of a finite duration.

The constituent equations, in the linear response approximation and for cubic solids or isotropic systems (no tensor structure), are

$$\vec{\widetilde{D}}\,(\vec{r},\omega) = \widetilde{\epsilon}\,(\omega)\ \ \vec{\widetilde{E}}\,(\vec{r},\omega)\ , \tag{1.133a}$$

$$\vec{\widetilde{H}}\,(\vec{r},\omega) = \widetilde{\mu}\,(\omega)^{-1}\ \ \vec{\widetilde{B}}\,(\vec{r},\omega)\ . \tag{1.133b}$$

The permittivity $\widetilde{\epsilon}\,(\omega)$ and the permeability $\widetilde{\mu}\,(\omega)$ are complex functions (have an imaginary part) if the medium is dispersive (i.e., if there is an absorption process). The charge conservation law is

$$\vec{\nabla}\cdot\vec{J}\,(\vec{r},t) + \frac{\partial}{\partial t}\rho\,(\vec{r},t) = 0\,, \qquad \vec{\nabla}\cdot\vec{\widetilde{J}}\,(\vec{r},\omega) - \mathrm{i}\,\omega\,\widetilde{\rho}\,(\vec{r},\omega) = 0\,. \tag{1.134}$$

The condition that the time dependent fields and sources be real, places constraints on the Fourier transformed fields and sources. We assume that $F\,(t)$ is a real function and $\widetilde{F}\,(\omega)$ is its Fourier transform. Then,

$$\widetilde{F}\,(\omega) = \int_{-\infty}^{\infty} F\,(t)\ \exp\left(\mathrm{i}\,\omega\,t\right)\,\mathrm{d}t\,. \tag{1.135}$$

The Fourier transform $\widetilde{F}(\omega)$ generally is a complex function. Furthermore, the physical dimensions of $F(t)$ and $\widetilde{F}(\omega)$ are not equal to each other: The function $\widetilde{F}(\omega)$ is obtained from $F(t)$ by an integration over time, which implies that $\widetilde{F}(\omega)$ has the dimension of $F(t)$, multiplied by the physical dimension of time (SI mksA unit: second). The Fourier transform of the electric field, $\vec{\widetilde{E}}(\vec{r},\omega)$, thus has SI mksA units of $\mathrm{V\,s/m}$.

If $F(t)$ is a real function, then its complex conjugate can be written as

$$\widetilde{F}(\omega)^* = \int_{-\infty}^{\infty} \mathrm{d}t\; F(t)\; \exp(-\mathrm{i}\,\omega\, t) = F(-\omega)\,. \tag{1.136}$$

We can thus establish that

$$F(t)\ \text{real} \quad \Leftrightarrow \quad \widetilde{F}(\omega)^* = \widetilde{F}(-\omega)\,. \tag{1.137}$$

The real and imaginary parts of $\widetilde{F}$ can thus be written as

$$\mathrm{Re}\,\widetilde{F}(\omega) = \frac{1}{2}\left\{\widetilde{F}(\omega) + \widetilde{F}(\omega)^*\right\} = \frac{1}{2}\left\{\widetilde{F}(\omega) + \widetilde{F}(-\omega)\right\}, \quad \mathrm{Re}\,\widetilde{F}(\omega) \;=\; \mathrm{Re}\,\widetilde{F}(-\omega)\,, \tag{1.138a}$$

$$\mathrm{Im}\,\widetilde{F}(\omega) = \frac{1}{2\mathrm{i}}\left\{\widetilde{F}(\omega) - \widetilde{F}(\omega)^*\right\} = \frac{1}{2\mathrm{i}}\left\{\widetilde{F}(\omega) - \widetilde{F}(-\omega)\right\}, \quad \mathrm{Im}\,\widetilde{F}(\omega) \;=\; -\mathrm{Im}\,\widetilde{F}(-\omega)\,. \tag{1.138b}$$

So: If $F(t)$ is real, then the real part of $\widetilde{F}(\omega)$ is an even function of ω, whereas the imaginary part of $\widetilde{F}(\omega)$ is an odd function of ω. The integral norm is preserved under the Fourier transformation,

$$||F||^2 = \int_{-\infty}^{\infty} \mathrm{d}t\;\, |F(t)|^2 = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\;\left|\widetilde{F}(\omega)\right|^2\,. \tag{1.139}$$

Even if $F(t)$ is not strictly real, this can be easily shown:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\;\left|\widetilde{F}(\omega)\right|^2 &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\;\widetilde{F}(\omega)\;\widetilde{F}^*(\omega) \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\;\int_{-\infty}^{\infty} \mathrm{d}t'\,\mathrm{e}^{\mathrm{i}\,\omega\,t'}\,F(t')\int_{-\infty}^{\infty}\mathrm{d}t''\,\mathrm{e}^{-\mathrm{i}\,\omega\,t''}\,F^*(t'') \\
&= \int_{-\infty}^{\infty} \mathrm{d}t'\;\int_{-\infty}^{\infty} \mathrm{d}t''\;\;\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\;\mathrm{e}^{\mathrm{i}\,\omega\,(t'-t'')}\,F(t')\,F^*(t'') \\
&= \int_{-\infty}^{\infty} \mathrm{d}t'\;\int_{-\infty}^{\infty} \mathrm{d}t''\;\;\delta(t'-t'')\,F(t')\,F^*(t'') = \int_{-\infty}^{\infty}\mathrm{d}t'\;|F(t')|^2\,.
\end{aligned} \tag{1.140}$$

It follows that, if $F(t)$ is real, then

$$\langle\omega\rangle = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\;\omega\;\left|\widetilde{F}(\omega)\right|^2 = 0\,. \tag{1.141}$$

because $\left|\widetilde{F}(\omega)\right| = \left|\widetilde{F}^*(\omega)\right| = \left|\widetilde{F}(-\omega)\right|$.

For general $\widetilde{F}(\omega)$, the normalized expectation value, or mean value, of the frequency and its square are given by

$$\langle\omega\rangle = \frac{1}{||F||^2}\;\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\;\omega\;\left|\widetilde{F}(\omega)\right|^2 = \frac{1}{||F||^2}\;\int_{-\infty}^{\infty}\mathrm{d}t\;\;F^*(t)\left(\mathrm{i}\frac{\partial}{\partial t}\right)\,F(t)\,, \tag{1.142a}$$

$$\left\langle\omega^2\right\rangle = \frac{1}{||F||^2}\;\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\;\omega^2\;\left|\widetilde{F}(\omega)\right|^2 = \frac{1}{||F||^2}\;\int_{-\infty}^{\infty}\mathrm{d}t\;F^*(t)\left(-\frac{\partial^2}{\partial t^2}\right)\,F(t)\,. \tag{1.142b}$$

The frequency width then is given by

$$\Delta\omega^2 = \left\langle (\omega - \langle\omega\rangle)^2 \right\rangle = \frac{1}{||F||^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \, (\omega - \langle\omega\rangle)^2 \, \left|\widetilde{F}(\omega)\right|^2$$
$$= \left\langle \omega^2 \right\rangle - 2 \langle \omega \langle\omega\rangle\rangle + \langle\omega\rangle^2 = \left\langle\omega^2\right\rangle - \langle\omega\rangle^2 \, . \tag{1.143}$$

Analogous relations hold for the temporal expectation values,

$$\langle t \rangle = \frac{1}{||F||^2} \int_{-\infty}^{\infty} \mathrm{d}t \; t \; |F(t)|^2 = \frac{1}{||F||^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \, F^*(\omega) \left(-\mathrm{i}\frac{\partial}{\partial\omega}\right) F(\omega) \, , \tag{1.144a}$$

$$\left\langle t^2 \right\rangle = \frac{1}{||F||^2} \int_{-\infty}^{\infty} \mathrm{d}t \; t^2 \; |F(t)|^2 = \frac{1}{||F||^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \, F^*(\omega) \left(-\frac{\partial^2}{\partial\omega^2}\right) F(\omega) \, . \tag{1.144b}$$

The temporal width is given by

$$\Delta t^2 = \left\langle (t - \langle t\rangle)^2 \right\rangle = \frac{1}{||F||^2} \int_{-\infty}^{\infty} \mathrm{d}t \; (t - \langle t\rangle)^2 \; |F(t)|^2 = \left\langle t^2 \right\rangle - \langle t\rangle^2 \, . \tag{1.145}$$

With $\Delta t = \sqrt{\Delta t^2}$ and $\Delta\omega = \sqrt{\Delta\omega^2}$, a useful relation is

$$\Delta t \cdot \Delta\omega \gtrsim \tfrac{1}{2} \, . \tag{1.146}$$

This relationship is given here without proof; it relates the frequency and temporal widths of the fields. This relationship is similar to the Heisenberg uncertainty relation obtained in ordinary quantum mechanics that holds for matter waves.

Let us now apply the wisdom learned to the dielectric displacement and to the electric field. Since $\vec{E}$ and $\vec{B}$ are real, we have for the Fourier transforms

$$\mathrm{Re}[\tilde{\vec{E}}(\vec{r},\omega)] = \mathrm{Re}[\tilde{\vec{E}}(\vec{r},-\omega)] \, , \qquad \mathrm{Im}[\tilde{\vec{E}}(\vec{r},\omega)] = -\mathrm{Im}[\tilde{\vec{E}}(\vec{r},-\omega)] \, , \tag{1.147a}$$

$$\mathrm{Re}[\tilde{\vec{B}}(\vec{r},\omega)] = \mathrm{Re}[\tilde{\vec{B}}(\vec{r},-\omega)] \, , \qquad \mathrm{Im}[\tilde{\vec{B}}(\vec{r},\omega)] = -\mathrm{Im}[\tilde{\vec{B}}(\vec{r},-\omega)] \, . \tag{1.147b}$$

The constituent equations in Fourier space are

$$\tilde{\vec{D}}(\vec{r},\omega) = \widetilde{\epsilon}(\omega) \; \tilde{\vec{E}}(\vec{r},\omega) \, , \qquad \tilde{\vec{B}}(\vec{r},\omega) = \widetilde{\mu}(\omega) \; \tilde{\vec{H}}(\vec{r},\omega) \, , \tag{1.148}$$

which translates into the following equations in coordinate space,

$$\tilde{\vec{D}}(\vec{r},t) = \int \mathrm{d}t' \; \epsilon(t-t') \; \tilde{\vec{E}}(\vec{r},t') \, , \qquad \tilde{\vec{B}}(\vec{r},t) = \int \mathrm{d}t' \; \mu(t-t') \; \tilde{\vec{H}}(\vec{r},t') \, . \tag{1.149}$$

In contrast to the literature, we here denote the Fourier transforms of the permittivity and permeability by the different symbols $\widetilde{\epsilon}$ and $\widetilde{\mu}$, where

$$\epsilon(t-t') = \int \frac{\mathrm{d}\omega}{2\pi} \, \widetilde{\epsilon}(\omega) \; \mathrm{e}^{-\mathrm{i}\,\omega\,(t-t')} \, , \qquad \mu(t-t') = \int \frac{\mathrm{d}\omega}{2\pi} \, \widetilde{\mu}(\omega) \; \mathrm{e}^{-\mathrm{i}\,\omega\,(t-t')} \, . \tag{1.150}$$

If $\widetilde{\epsilon}(\omega)$ is a function of a single argument in frequency space and leads to a convolution in time, we can give the following physical interpretation: The system is modelled as a collection of anharmonic oscillators which can be driven on or off resonance. The (damped) oscillators correspond to the atomic transitions in the material. In space-time, these oscillations can continue long after the electromagnetic pulse train has passed: hence, the need for the evaluation of convolution integrals in time. Now, we can project the dielectric displacement $\vec{D}$ and the magnetic field $\vec{H}$ onto their real and imaginary parts,

$$\mathrm{Re}[\tilde{\vec{D}}(\vec{r},\omega)] = \mathrm{Re}[\widetilde{\epsilon}(\omega)]\ \mathrm{Re}[\tilde{\vec{E}}(\vec{r},\omega)] - \mathrm{Im}[\widetilde{\epsilon}(\omega)]\ \mathrm{Im}[\tilde{\vec{E}}(\vec{r},\omega)]\,, \tag{1.151a}$$

$$\mathrm{Im}[\tilde{\vec{D}}(\vec{r},\omega)] = \mathrm{Re}[\widetilde{\epsilon}(\omega)]\ \mathrm{Im}[\tilde{\vec{E}}(\vec{r},\omega)] + \mathrm{Im}[\widetilde{\epsilon}(\omega)]\ \mathrm{Re}[\tilde{\vec{E}}(\vec{r},\omega)]\,. \tag{1.151b}$$

If now $\widetilde{\epsilon}(\omega)$ and $\widetilde{\mu}(\omega)$ have the properties

$$\mathrm{Re}[\widetilde{\epsilon}(\omega)] = \mathrm{Re}[\widetilde{\epsilon}(-\omega)]\,, \qquad \mathrm{Im}[\widetilde{\epsilon}(\omega)] = -\mathrm{Im}[\widetilde{\epsilon}(-\omega)]\,, \tag{1.152a}$$

$$\mathrm{Re}[\widetilde{\mu}(\omega)] = \mathrm{Re}[\widetilde{\mu}(-\omega)]\,, \qquad \mathrm{Im}[\widetilde{\mu}(\omega)] = -\mathrm{Im}[\widetilde{\mu}(-\omega)]\,, \tag{1.152b}$$

[the same as for the $\vec{E}$ and $\vec{B}$ fields], then

$$\mathrm{Re}[\tilde{\vec{D}}(\vec{r},\omega)] = \mathrm{Re}[\tilde{\vec{D}}(\vec{r},-\omega)]\,, \qquad \mathrm{Im}[\tilde{\vec{D}}(\vec{r},\omega)] = -\mathrm{Im}[\tilde{\vec{D}}(\vec{r},-\omega)]\,, \tag{1.153a}$$

$$\mathrm{Re}[\tilde{\vec{H}}(\vec{r},\omega)] = \mathrm{Re}[\tilde{\vec{H}}(\vec{r},-\omega)]\,, \qquad \mathrm{Im}[\tilde{\vec{H}}(\vec{r},\omega)] = -\mathrm{Im}[\tilde{\vec{H}}(\vec{r},-\omega)]\,. \tag{1.153b}$$

The latter equations ensure that the $\vec{D}$ and $\vec{H}$ fields are real.

1.3.2 Dissipation and Energy Storage for Pulse Mode

Fourier integrals have a common characteristic feature: They are well defined only if the input function is adiabatically damped in the infinite past and future. Therefore, we first study pulse mode where a pulse train of finite duration passes a well-defined spatial region. Our observations of a system and its interaction with an electromagnetic field are made over time intervals. The observation time interval measured in the characteristic times of the fields and the system can range from very small to very large. At this point we will consider the time interval for measurement to be large compared with the characteristic times of the system. The electromagnetic fields can either be cw or pulse. In the case of cw operation, we generally assume that the system is in the "steady state," and our measurements are time averages of the system and field properties. In the pulse mode, we measure the average properties of the system and the fields per pulse.

We recall from Sec. 1.2.3 the Poynting vector,

$$\vec{S}\,(\vec{r},t) = \vec{E}\,(\vec{r},t) \times \vec{H}\,(\vec{r},t)\ , \tag{1.154}$$

the (time derivative of the) energy density of the electromagnetic field,

$$\frac{\partial}{\partial t}u\,(\vec{r},t) = \frac{1}{2}\left[\vec{E}(\vec{r},t)\cdot\frac{\partial}{\partial t}\vec{D}(\vec{r},t) + \vec{H}\,(\vec{r},t)\cdot\frac{\partial}{\partial t}\vec{B}\,(\vec{r},t)\right]\ , \tag{1.155}$$

the local power density (characterizing the mechanical power density for the work done on the sample),

$$\frac{\partial P\,(\vec{r},t)}{\partial V} = \vec{E}\,(\vec{r},t)\cdot\vec{J}_0\,(\vec{r},t)\ . \tag{1.156}$$

Poynting's theorem, in differential form, then reads

Poynting's Theorem:
$$\vec{\nabla}\cdot\vec{S}(\vec{r},t) = -\frac{\partial u\,(\vec{r},t)}{\partial t} - \frac{\partial P\,(\vec{r},t)}{\partial V} = -\frac{\partial u\,(\vec{r},t)}{\partial t} - \vec{E}\,(\vec{r},t)\cdot\vec{J}_0\,(\vec{r},t)\ . \tag{1.157}$$

In comparison to Eq. (1.53), we here assume that $\vec{J}_0\,(\vec{r},t)$ is only the *free* current density.

A phenomenological interpretation is this: Imagine the integrated version of this theorem, over a small sample volume δV. The divergence of the Poynting vector $\vec{S}$ measures the rate at which energy density leaves the sample volume. If this divergence is positive, then field energy leaves the sample volume. This can be compensated by a loss in the energy density, with a negative time derivative of u, leading to a positive time derivative $-\partial_t u > 0$. Or, work is done on the charges in the sample volume, against the electric field, resulting in a free current density which fulfills $-\vec{E}\cdot\vec{J}_0 > 0$, because $\vec{E}$ is antiparallel with respect to $\vec{J}_0$. Here, work is done on the moving charges, against the direction of the electric field which would otherwise propel the charges in the other direction.

Integrating the Poynting theorem with respect to space, we have

$$\int_{\partial V} \vec{S}(\vec{r},t)\cdot \mathrm{d}\vec{A} = -\int_V \vec{E}(\vec{r},t)\cdot\vec{J}_0(\vec{r},t)\,\mathrm{d}^3 r - \int_V \frac{\partial u\,(\vec{r},t)}{\partial t}\,\mathrm{d}^3 r\,. \tag{1.158}$$

However, we can also integrate the differential form with respect to time,

$$\int \mathrm{d}t\,\vec{\nabla}\cdot\vec{S}(\vec{r},t) = -\int \mathrm{d}t\,\frac{\partial u\,(\vec{r},t)}{\partial t} - \int \mathrm{d}t\,\vec{E}\,(\vec{r},t)\cdot\vec{J}_0\,(\vec{r},t)\;. \tag{1.159}$$

We here consider the case of the pulse mode. We assume that there is a laser pulse of finite duration. So, all integrals taken from $t=-\infty$ to $t=+\infty$ must necessarily be finite, which defines an important assumption made in the following discussion. It is instructive to convert all relevant expressions to Fourier space (with respect to time). The power density delivered to the current by the fields, integrated over all time (thus, equal to the energy density) is

$$\begin{aligned}
w\,(\vec{r}) &= \int_{-\infty}^{\infty}\mathrm{d}t\,\vec{E}\,(\vec{r},t)\cdot\vec{J}_0\,(\vec{r},t) \\
&= \int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{2\pi}\,\vec{\tilde{E}}\,(\vec{r},\omega)\cdot\vec{\tilde{J}}_0\,(\vec{r},\omega')\exp\left[-\mathrm{i}\,(\omega+\omega')\;t\right] \\
&= \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{2\pi}\,\vec{\tilde{E}}\,(\vec{r},\omega)\cdot\vec{\tilde{J}}_0\,(\vec{r},\omega')\;2\pi\delta(\omega+\omega') \\
&= \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\,\vec{\tilde{E}}\,(\vec{r},-\omega)\cdot\vec{\tilde{J}}_0\,(\vec{r},\omega) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\,\vec{\tilde{E}}\,(\vec{r},\omega)^*\cdot\vec{\tilde{J}}_0\,(\vec{r},\omega)\;.
\end{aligned} \tag{1.160}$$

The energy density going to the "fields" from the infinite past to the infinite future, is given by the integral over the time derivative of the power density $u(\vec{r},t)$ stored in the fields,

$$\begin{aligned}
u(\vec{r}) &= \int_{-\infty}^{\infty}\mathrm{d}t\,\frac{\partial}{\partial t}u(\vec{r},t) = 2\times\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}t\,\left[\vec{\tilde{H}}\,(\vec{r},t)\cdot\frac{\partial}{\partial t}\vec{\tilde{B}}\,(\vec{r},t)+\vec{\tilde{E}}\,(\vec{r},t)\cdot\frac{\partial}{\partial t}\vec{\tilde{D}}\,(\vec{r},t)\right] \\
&= \int_{-\infty}^{\infty}\mathrm{d}t\,\left[\int\frac{\mathrm{d}\omega}{2\pi}\,\vec{\tilde{H}}\,(\vec{r},\omega)\;\mathrm{e}^{-\mathrm{i}\,\omega\,t}\cdot\frac{\partial}{\partial t}\int\frac{\mathrm{d}\omega'}{2\pi}\,\vec{\tilde{B}}\,(\vec{r},\omega')\;\mathrm{e}^{-\mathrm{i}\,\omega'\,t}\right. \\
&\qquad \left.+\int\frac{\mathrm{d}\omega}{2\pi}\,\vec{\tilde{E}}\,(\vec{r},\omega)\;\mathrm{e}^{-\mathrm{i}\,\omega\,t}\cdot\frac{\partial}{\partial t}\int\frac{\mathrm{d}\omega'}{2\pi}\,\vec{\tilde{D}}\,(\vec{r},\omega')\;\mathrm{e}^{-\mathrm{i}\,\omega'\,t}\right] \\
&= \int_{-\infty}^{\infty}\mathrm{d}t\,\left[\int\frac{\mathrm{d}\omega}{2\pi}\,\vec{\tilde{H}}\,(\vec{r},\omega)\;\mathrm{e}^{-\mathrm{i}\,\omega\,t}\cdot\int\frac{\mathrm{d}\omega'}{2\pi}\,(-\mathrm{i}\omega')\,\vec{\tilde{B}}\,(\vec{r},\omega')\;\mathrm{e}^{-\mathrm{i}\,\omega'\,t}\right. \\
&\qquad \left.+\int\frac{\mathrm{d}\omega}{2\pi}\,\vec{\tilde{E}}\,(\vec{r},\omega)\;\mathrm{e}^{-\mathrm{i}\,\omega\,t}\int\frac{\mathrm{d}\omega'}{2\pi}\,(-\mathrm{i}\omega')\,\vec{\tilde{D}}\,(\vec{r},\omega')\;\mathrm{e}^{-\mathrm{i}\,\omega'\,t}\right]\,.
\end{aligned} \tag{1.161}$$

We can now do the integrals over ω and write

$$
\begin{aligned}
u(\vec{r}) &= \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} \, 2\pi\delta(\omega+\omega') \, (-\mathrm{i}\omega') \left[\tilde{\vec{H}}(\vec{r},\omega) \, \tilde{\vec{B}}(\vec{r},\omega') + \tilde{\vec{E}}(\vec{r},\omega) \;\; \tilde{\vec{D}}(\vec{r},\omega') \right] \\
&= \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} \, 2\pi\delta(\omega+\omega') \, (-\mathrm{i}\omega') \, \tilde{\vec{H}}(\vec{r},\omega) \, \tilde{\vec{B}}(\vec{r},\omega') \\
&\quad + \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} \, (-\mathrm{i}\omega') \, 2\pi\delta(\omega+\omega') \, \tilde{\vec{E}}(\vec{r},\omega) \;\; \tilde{\vec{D}}(\vec{r},\omega') \\
&= \int \frac{\mathrm{d}\omega'}{2\pi} \, (+\mathrm{i}\omega) \, \tilde{\vec{H}}(\vec{r},\omega) \, \tilde{\vec{B}}(\vec{r},-\omega) + \int \frac{\mathrm{d}\omega}{2\pi} \, (+\mathrm{i}\omega) \, \tilde{\vec{E}}(\vec{r},\omega) \;\; \tilde{\vec{D}}(\vec{r},-\omega) \\
&= \mathrm{i} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \, \omega \left[\tilde{\vec{H}}(\vec{r},\omega) \cdot \tilde{\vec{B}}(\vec{r},\omega)^* + \tilde{\vec{E}}(\vec{r},\omega) \cdot \tilde{\vec{D}}(\vec{r},\omega)^* \right] . \qquad (1.162)
\end{aligned}
$$

We might have carried out the integrations in a different orders, resulting in the expression

$$
\begin{aligned}
u(\vec{r}) &= \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} \, 2\pi\delta(\omega+\omega') \, (-\mathrm{i}\omega') \left[\tilde{\vec{H}}(\vec{r},\omega) \, \tilde{\vec{B}}(\vec{r},\omega') + \tilde{\vec{E}}(\vec{r},\omega) \;\; \tilde{\vec{D}}(\vec{r},\omega') \right] \\
&= \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} \, 2\pi\delta(\omega+\omega') \, (-\mathrm{i}\omega') \, \tilde{\vec{H}}(\vec{r},\omega) \, \tilde{\vec{B}}(\vec{r},\omega') \\
&\quad + \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} \, (-\mathrm{i}\omega') \, 2\pi\delta(\omega+\omega') \, \tilde{\vec{E}}(\vec{r},\omega) \;\; \tilde{\vec{D}}(\vec{r},\omega') \\
&= \int \frac{\mathrm{d}\omega'}{2\pi} \, (-\mathrm{i}\omega') \, \tilde{\vec{H}}(\vec{r},-\omega') \, \tilde{\vec{B}}(\vec{r},\omega') + \int \frac{\mathrm{d}\omega}{2\pi} \, (+\mathrm{i}\omega) \, \tilde{\vec{E}}(\vec{r},\omega) \;\; \tilde{\vec{D}}(\vec{r},-\omega) \\
&= \mathrm{i} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \, \omega \left[-\tilde{\vec{H}}(\vec{r},\omega)^* \cdot \tilde{\vec{B}}(\vec{r},\omega) + \tilde{\vec{E}}(\vec{r},\omega) \cdot \tilde{\vec{D}}(\vec{r},\omega)^* \right] . \qquad (1.163)
\end{aligned}
$$

This differs from the result given in Eq. (1.162) by the sign of the first term in square brackets (and by the quantity which is complex conjugated). We shall use the form (1.162) in the following, keeping in mind that a form analogous to Eq. (1.163) may be useful later (in the analysis of the cw mode).

The energy density carried by the Poynting vector, integrated over all time, is

$$
\begin{aligned}
\vec{S}(\vec{r}) &= \int_{-\infty}^{\infty} \mathrm{d}t \, \vec{S}(\vec{r},t) = \int_{-\infty}^{\infty} \mathrm{d}t \left[\tilde{\vec{E}}(\vec{r},t) \times \tilde{\vec{H}}(\vec{r},t) \right] \\
&= \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{2\pi} \left(\tilde{\vec{E}}(\vec{r},\omega) \;\, \mathrm{e}^{-\mathrm{i}\,\omega\, t} \right) \times \left(\tilde{\vec{H}}(\vec{r},\omega') \;\, \mathrm{e}^{-\mathrm{i}\,\omega'\, t} \right) \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{2\pi} \, 2\pi\,\delta(\omega+\omega') \left(\tilde{\vec{E}}(\vec{r},\omega) \;\times \tilde{\vec{H}}(\vec{r},\omega') \right) \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \left[\tilde{\vec{E}}(\vec{r},\omega) \times \tilde{\vec{H}}(\vec{r},-\omega) \right] = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \left[\tilde{\vec{E}}(\vec{r},\omega) \times \tilde{\vec{H}}(\vec{r},\omega)^* \right] . \qquad (1.164)
\end{aligned}
$$

The integrated form of Poynting's theorem (with respect to time, not space) reads as

$$
\int \mathrm{d}t \, \vec{\nabla} \cdot \vec{S}(\vec{r},t) + \int \mathrm{d}t \, \frac{\partial u(\vec{r},t)}{\partial t} + \int \mathrm{d}t \, \vec{E}(\vec{r},t) \cdot \vec{J}_0(\vec{r},t) = 0 \, . \qquad (1.165)
$$

It can be expressed in terms of the Fourier-transformed quantities just calculated,

Integrated Poynting's Theorem: $\quad \vec{\nabla} \cdot \vec{S}(\vec{r}) + u(\vec{r}) + w(\vec{r}) = 0 \, . \qquad (1.166)$

We recall that

$$\vec{S}\,(\vec{r}) = \int_{-\infty}^{\infty} \mathrm{d}t\, \left[\vec{E}\,(\vec{r},t) \times \vec{H}\,(\vec{r},t)\right] = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \left[\vec{\tilde{E}}\,(\vec{r},\omega) \times \vec{\tilde{H}}\,(\vec{r},\omega)^*\right] , \tag{1.167a}$$

$$\begin{aligned} u(\vec{r}) &= \int_{-\infty}^{\infty} \mathrm{d}t\, \left[\vec{H}\,(\vec{r},t) \cdot \frac{\partial}{\partial t}\vec{B}\,(\vec{r},t) + \vec{E}\,(\vec{r},t) \cdot \frac{\partial}{\partial t}\vec{D}\,(\vec{r},t)\right] \\ &= \mathrm{i}\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\,\omega\, \left[\vec{\tilde{H}}\,(\vec{r},\omega) \cdot \vec{\tilde{B}}\,(\vec{r},\omega)^* + \vec{\tilde{E}}\,(\vec{r},\omega) \cdot \vec{\tilde{D}}\,(\vec{r},\omega)^*\right] , \end{aligned} \tag{1.167b}$$

$$w\,(\vec{r}) = \int_{-\infty}^{\infty} \mathrm{d}t\, \vec{E}\,(\vec{r},t) \cdot \vec{J}_0\,(\vec{r},t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \vec{\tilde{E}}\,(\vec{r},\omega)^* \cdot \vec{\tilde{J}}_0\,(\vec{r},\omega)\ . \tag{1.167c}$$

We thus have

$$\begin{aligned} &\vec{\nabla} \cdot \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \left[\vec{\tilde{E}}\,(\vec{r},\omega) \times \vec{\tilde{H}}\,(\vec{r},\omega)^*\right] + \mathrm{i}\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\,\omega\, \left[\vec{\tilde{H}}\,(\vec{r},\omega) \cdot \vec{\tilde{B}}\,(\vec{r},\omega)^* + \vec{\tilde{E}}\,(\vec{r},\omega) \cdot \vec{\tilde{D}}\,(\vec{r},\omega)^*\right] \\ &\quad + \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \vec{\tilde{E}}\,(\vec{r},\omega)^* \cdot \vec{\tilde{J}}_0\,(\vec{r},\omega)\ =\ 0\,. \end{aligned} \tag{1.168}$$

We now consider under what conditions there will be a net transfer to or from the fields during a pulse, i.e., under what conditions

$$u(\vec{r}) \neq 0\,, \qquad \text{or merely} \qquad u(\vec{r}) > 0\,. \tag{1.169}$$

We shall see that by virtue of the properties of the Fourier transforms of the fields just shown, we need nonvanishing imaginary parts for the permittivity and permeability functions $\epsilon(\omega)$ and $\widetilde{\mu}(\omega)$ in order for the energy dissipation to be nonvanishing. For this we use the constituent equations to rewrite $u\,(\vec{r})$ in terms of $\vec{\tilde{E}}\,(\vec{r},\omega)$ and $\vec{\tilde{H}}\,(\vec{r},\omega)$, and we also use

$$\left|\vec{\tilde{H}}\,(\vec{r},\omega)\right|^2 = \left|\vec{\tilde{H}}\,(\vec{r},-\omega)\right|^2, \qquad \left|\vec{\tilde{D}}\,(\vec{r},\omega)\right|^2 = \left|\vec{\tilde{D}}\,(\vec{r},-\omega)\right|^2. \tag{1.170}$$

The energy dissipated into the field can thus be written as

$$\begin{aligned} u\,(\vec{r}) &= \mathrm{i}\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\,\omega\, \left[\vec{\tilde{H}}\,(\vec{r},\omega) \cdot \vec{\tilde{B}}\,(\vec{r},\omega)^* + \vec{\tilde{E}}\,(\vec{r},\omega) \cdot \vec{\tilde{D}}\,(\vec{r},\omega)^*\right] \\ &= \mathrm{i}\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\,\omega\, \left[\widetilde{\mu}\,(\omega)^*\, \left|\vec{\tilde{H}}\,(\vec{r},\omega)\right|^2 + \widetilde{\epsilon}\,(\omega)^*\, \left|\vec{\tilde{E}}\,(\vec{r},\omega)\right|^2\right] \\ &= \mathrm{i}\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\,\omega\, \left[\mathrm{Re}[\widetilde{\mu}\,(\omega)]\, \left|\vec{\tilde{H}}\,(\vec{r},\omega)\right|^2 + \mathrm{Re}[\widetilde{\epsilon}\,(\omega)]\, \left|\vec{\tilde{E}}\,(\vec{r},\omega)\right|^2\right] \\ &\quad + \mathrm{i}\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\,\omega\, \left[(-\mathrm{i}\,\mathrm{Im}[\widetilde{\mu}\,(\omega)])\, \left|\vec{\tilde{H}}\,(\vec{r},\omega)\right|^2 + (-\mathrm{i}\,\mathrm{Im}[\widetilde{\epsilon}\,(\omega)])\, \left|\vec{\tilde{E}}\,(\vec{r},\omega)\right|^2\right] \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\,\omega\, \left[\mathrm{Im}[\widetilde{\mu}\,(\omega)]\, \left|\vec{\tilde{H}}\,(\vec{r},\omega)\right|^2 + \mathrm{Im}[\widetilde{\epsilon}\,(\omega)]\, \left|\vec{\tilde{E}}\,(\vec{r},\omega)\right|^2\right] , \end{aligned} \tag{1.171}$$

because the integral with the real parts vanishes in view of the symmetry relations (1.147), (1.153), and (1.152). Thus, $u\,(\vec{r})$ is non-zero only if the imaginary parts of μ and ϵ do not both vanish. In this case, energy will be dissipated in the response of the mechanical system to the applied fields. If $\widetilde{\mu}\,(\omega)$ and $\widetilde{\epsilon}\,(\omega)$ are both real, the system is not dispersive and no energy is lost doing mechanical work on the sources.

From now on, we shall denote the Fourier transforms of ϵ and μ by the same symbols; it would be somewhat pedantic to keep the distinction in the following sections and chapters. This somewhat ambiguous notation can otherwise be justified on a different basis: Namely, we can "overload" (in the sense of modular programming, see Fortran90 syntax) the symbols ϵ and μ by different functions, depending on the physical dimension of the argument. So, if the argument of ϵ has the dimension of frequency, then we would "call" the function previously denoted by $\widetilde{\epsilon}$, whereas if the argument of ϵ has the dimension of time, then we would "call" the function previously denoted by ϵ.

Let us consider the alternative formulation from Eq. (1.163) and calculate the dissipated energy density as follows,

$$\begin{aligned}
u\,(\vec{r}) &= \mathrm{i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\,\omega\,\left[-\vec{\widetilde{H}}\,(\vec{r},\omega)^*\cdot\vec{\widetilde{B}}\,(\vec{r},\omega)+\vec{\widetilde{E}}\,(\vec{r},\omega)\cdot\vec{\widetilde{D}}\,(\vec{r},\omega)^*\right] \\
&= \mathrm{i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\,\omega\,\left[-\frac{1}{\widetilde{\mu}\,(\omega)^*}\left|\vec{\widetilde{B}}\,(\vec{r},\omega)\right|^2+\widetilde{\epsilon}\,(\omega)^*\left|\vec{\widetilde{E}}\,(\vec{r},\omega)\right|^2\right] \\
&= \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\,\omega\,\left[\frac{\mathrm{Im}[\widetilde{\mu}\,(\omega)]}{|\widetilde{\mu}(\omega)|^2}\left|\vec{\widetilde{H}}\,(\vec{r},\omega)\right|^2+\mathrm{Im}[\widetilde{\epsilon}\,(\omega)]\left|\vec{\widetilde{E}}\,(\vec{r},\omega)\right|^2\right]\,. \qquad (1.172)
\end{aligned}$$

Causality dictates, as we will see later, that $\mathrm{Im}[\widetilde{\mu}\,(\omega)]$ and $\mathrm{Im}[\widetilde{\epsilon}\,(\omega)]$ are positive for positive driving frequencies, and so the inequality $u\,(\vec{r})\geq 0$ is fulfilled, as it should be for an absorptive process.

An important remark: The time integrals in expressions like

$$\begin{aligned}
u(\vec{r}) &= \int_{-\infty}^{\infty}\mathrm{d}t\,\left[\vec{H}\,(\vec{r},t)\cdot\frac{\partial}{\partial t}\vec{B}\,(\vec{r},t)+\vec{E}\,(\vec{r},t)\cdot\frac{\partial}{\partial t}\vec{D}\,(\vec{r},t)\right] \\
&= \mathrm{i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\,\omega\,\left[\vec{H}\,(\vec{r},\omega)\cdot\vec{B}\,(\vec{r},\omega)^*+\vec{E}\,(\vec{r},\omega)\cdot\vec{D}\,(\vec{r},\omega)^*\right]\,, \qquad (1.173)
\end{aligned}$$

are convergent only if the fields are adiabatically damped in the distant past and future. This includes the pulsed mode and excludes the cw mode as a possible means of describing our system. For cw mode, a priori, the fields are not adiabatically damped in the past and future.

1.3.3 Dissipation and Energy Storage for Continuous Wave Mode

In the continuous wave (cw) mode, there is a finite amount of energy delivered to the current per any time interval, and therefore, the total energy density (the power density integrated at a particular point in space, integrated over all time) diverges. For pulse mode, we had no trouble calculating the quantities that enter

$$\vec{\nabla}\cdot\vec{S}(\vec{r})+u(\vec{r})+w(\vec{r})=0\,, \qquad (1.174)$$

which are time integrals. For cw mode, however, all of these quantities will go to infinity,

$$\vec{S}\,(\vec{r}) = \int_{-\infty}^{\infty}\mathrm{d}t\,\left[\vec{E}\,(\vec{r},t)\times\vec{H}\,(\vec{r},t)\right]\to\infty \qquad (1.175a)$$

$$u(\vec{r}) = \int_{-\infty}^{\infty}\mathrm{d}t\,\left[\vec{H}\,(\vec{r},t)\cdot\frac{\partial}{\partial t}\vec{B}\,(\vec{r},t)+\vec{E}\,(\vec{r},t)\cdot\frac{\partial}{\partial t}\vec{D}\,(\vec{r},t)\right]\to\infty\,, \qquad (1.175b)$$

$$w\,(\vec{r}) = \int_{-\infty}^{\infty}\mathrm{d}t\,\vec{E}\,(\vec{r},t)\cdot\vec{J}_0\,(\vec{r},t)\to\infty\,. \qquad (1.175c)$$

A different approach is necessary for cw mode. Indeed, if the fields are periodic, then it becomes customary and advantageous to use complex fields, because only these can be broken down to unique frequencies in Fourier space. First, we observe that a function

$$\cos(\omega t) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}\right) = \mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}\omega t}\right), \tag{1.176}$$

has two frequency components at $\pm\omega$. An analogy with particle and antiparticle solutions might be discussed in the lecture. It turns out that the Maxwell equations hold for each frequency component $\pm\omega$ individually. Furthermore, because we can identify the physical field as the real part of the complex field proportional to $\mathrm{e}^{-\mathrm{i}\omega t}$, there is a strong indication to use a complex formalism with only one frequency component (Fourier component) altogether. In other words, we would consider the Maxwell equations for an individual Fourier component, which is canonically chosen as the Fourier component of angular frequency $+\omega$, i.e., proportional to $\exp(-\mathrm{i}\,\omega\,t)$ in time.

We can extend the analogy even further. Starting from a wave function proportional to

$$\cos(k\,x - \omega\,t) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}k\,x-\mathrm{i}\omega\,t} + \mathrm{e}^{-\mathrm{i}k\,x+\mathrm{i}\omega\,t}\right), \tag{1.177}$$

we obtain two complex waves with energy eigenvalues $\pm\omega$ of the time derivative operator $\mathrm{i}\partial_t$. Both of these are assigned to the same physical wave, proportional to $\cos(k\,x-\omega\,t)$, which moves into the positive x direction with wave vector k. (This consideration finds an application in the physical interpretation of antiparticle solutions of the Dirac equation, within quantum electrodynamics. Here, we only need this consideration in regard to the selection of the positive- or negative-frequency component of the electromagnetic fields. We choose the positive-frequency component, without loss of generality.)

So, if we assume that

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r})\,\mathrm{e}^{-\mathrm{i}\omega t}, \qquad \vec{B}(\vec{r},t) = \vec{B}(\vec{r})\,\mathrm{e}^{-\mathrm{i}\omega t}, \tag{1.178a}$$

$$\vec{D}(\vec{r},t) = \vec{D}(\vec{r})\,\mathrm{e}^{-\mathrm{i}\omega t}, \qquad \vec{H}(\vec{r},t) = \vec{H}(\vec{r})\,\mathrm{e}^{-\mathrm{i}\omega t}, \tag{1.178b}$$

$$\rho_0(\vec{r},t) = \rho_0(\vec{r})\,\mathrm{e}^{-\mathrm{i}\omega t}, \qquad \vec{J}_0(\vec{r},t) = \vec{J}_0(\vec{r})\,\mathrm{e}^{-\mathrm{i}\omega t}, \tag{1.178c}$$

then the Maxwell equations become

$$\begin{aligned} \vec{\nabla}\cdot\vec{D}(\vec{r}) &= \rho_0(\vec{r}), \qquad \vec{\nabla}\cdot\vec{B}(\vec{r}) = 0, \\ \vec{\nabla}\times\vec{E}(\vec{r}) - \mathrm{i}\omega\vec{B}(\vec{r}) &= \vec{0}, \qquad \vec{\nabla}\times\vec{H}(\vec{r}) + \mathrm{i}\omega\vec{D}(\vec{r}) = \vec{J}_0(\vec{r}). \end{aligned} \tag{1.179}$$

We now investigate the modified Poynting theorem implied by the complex quantities and write the work done on the sample volume V as

$$\begin{aligned} \vec{E}(\vec{r})\cdot\vec{J}_0^*(\vec{r}) &= \vec{E}(\vec{r})\cdot\left(\vec{\nabla}\times\vec{H}^*(\vec{r}) - \mathrm{i}\omega\vec{D}^*(\vec{r})\right) \\ &= \vec{E}(\vec{r})\cdot\left(\vec{\nabla}\times\vec{H}^*(\vec{r})\right) - \mathrm{i}\omega\vec{E}(\vec{r})\cdot\vec{D}^*(\vec{r}). \end{aligned} \tag{1.180}$$

Using the vector identity

$$\begin{aligned} \vec{\nabla}\cdot\left[\vec{E}(\vec{r})\times\vec{H}^*(\vec{r})\right] &= \vec{H}^*(\vec{r})\cdot\left(\vec{\nabla}\times\vec{E}(\vec{r})\right) - \vec{E}(\vec{r})\cdot\left(\vec{\nabla}\times\vec{H}^*(\vec{r})\right) \\ &= \vec{H}^*(\vec{r})\cdot\left(\mathrm{i}\omega\vec{B}(\vec{r})\right) - \vec{E}(\vec{r})\cdot\left(\vec{\nabla}\times\vec{H}^*(\vec{r})\right), \end{aligned} \tag{1.181}$$

we can derive that

$$\vec{E}(\vec{r})\cdot\left(\vec{\nabla}\times\vec{H}^*(\vec{r})\right) = -\vec{\nabla}\cdot\left[\vec{E}(\vec{r})\times\vec{H}^*(\vec{r})\right] + \mathrm{i}\omega\,\vec{B}(\vec{r})\cdot\vec{H}^*(\vec{r}). \tag{1.182}$$

Combining the above equations, we have

$$\vec{E}(\vec{r}) \cdot \vec{J}_0^*(\vec{r}) = -\vec{\nabla} \cdot \left[\vec{E}(\vec{r}) \times \vec{H}^*(\vec{r})\right] + \mathrm{i}\omega \, \vec{B}(\vec{r}) \cdot \vec{H}^*(\vec{r}) - \mathrm{i}\omega \, \vec{E}(\vec{r}) \cdot \vec{D}^*(\vec{r}) \,, \tag{1.183}$$

which can be written as

$$\tfrac{1}{2} \, \vec{\nabla} \cdot \left[\vec{E}(\vec{r}) \times \vec{H}^*(\vec{r})\right] + \tfrac{1}{2} \, \vec{E}(\vec{r}) \cdot \vec{J}_0^*(\vec{r}) + 2\mathrm{i}\omega \left(\tfrac{1}{4}\vec{E}(\vec{r}) \cdot \vec{D}^*(\vec{r}) - \tfrac{1}{4}\vec{B}(\vec{r}) \cdot \vec{H}^*(\vec{r})\right) = 0 \,. \tag{1.184}$$

We here identify the quantities (the brackets denotes the average over a single laser period)

$$\langle \vec{S}(\vec{r}) \rangle = \frac{1}{2}\vec{E}(\vec{r}) \times \vec{H}^*(\vec{r}) \,, \tag{1.185a}$$

$$\langle P(\vec{r}) \rangle = \tfrac{1}{2}\vec{E}(\vec{r}) \cdot \vec{J}_0^*(\vec{r}) \,, \tag{1.185b}$$

$$\langle w_e(\vec{r}) \rangle = \tfrac{1}{4}\vec{E}(\vec{r}) \cdot \vec{D}^*(\vec{r}) \,, \tag{1.185c}$$

$$\langle w_m(\vec{r}) \rangle = \tfrac{1}{4}\vec{B}(\vec{r}) \cdot \vec{H}^*(\vec{r}) \,. \tag{1.185d}$$

The pattern of complex conjugation is the same as adopted above in Eq. (1.171). So, the Poynting theorem becomes

$$\vec{\nabla} \cdot \langle \vec{S}(\vec{r}) \rangle + \langle P(\vec{r}) \rangle + 2\mathrm{i}\omega \left(\langle w_e(\vec{r}) \rangle - \langle w_m(\vec{r}) \rangle \right) = 0 \,. \tag{1.186}$$

This equation has a real and an imaginary part. First, let us assume a situation with $\vec{D}(\vec{r}) = \epsilon_0 \, \vec{E}(\vec{r})$ and $\vec{B}(\vec{r}) = \mu_0 \, \vec{H}(\vec{r})$. Then, the two quantities

$$\langle w_e(\vec{r}) \rangle = \frac{1}{4} \, \epsilon_0 \, \left|\vec{E}(\vec{r})\right|^2 \,, \qquad \langle w_m(\vec{r}) \rangle = \frac{1}{4} \, \frac{1}{\mu_0} \, \left|\vec{B}(\vec{r})\right|^2 \tag{1.187}$$

are manifestly real. The imaginary part of Eq. (1.186) then expresses the fact that for travelling waves, there is as much energy stored in the electric part of the wave as is stored in the magnetic part of the fields. By contrast, the real part of Eq. (1.186) expresses the fact that energy entering the volume (the negative volume integral of the time-averaged Poynting vector) manifests itself in work done on the sample (positive averaged power density $\langle P(\vec{r}) \rangle$).

For a general, dispersive medium, the electric part of the energy density is

$$\langle w_e(\vec{r}) \rangle = \frac{1}{4} \, \vec{E}(\vec{r}) \cdot \vec{D}^*(\vec{r}) = \frac{1}{4} \, \tilde{\epsilon}(\omega)^* \, \left|\vec{E}(\vec{r})\right|^2 \,, \tag{1.188}$$

and

$$\langle w_m(\vec{r}) \rangle = \frac{1}{4} \, \vec{B}(\vec{r}) \cdot \vec{H}^*(\vec{r}) = \frac{1}{4} \, \frac{1}{\tilde{\mu}(\omega)^*} \, \left|\vec{B}(\vec{r})\right|^2 \,. \tag{1.189}$$

If we adopt, for $\tilde{\epsilon}(\omega)$, our harmonic oscillator model and assume positive driving frequencies ω, we have

$$\mathrm{Im}[\tilde{\epsilon}^*(\omega)] = -\mathrm{Im}[\tilde{\epsilon}(\omega)] < 0 \,, \qquad \mathrm{Im}[\tilde{\mu}^*(\omega)] = -\mathrm{Im}[\tilde{\mu}(\omega)] < 0 \,, \qquad \mathrm{Im}\left[\frac{1}{\tilde{\mu}^*(\omega)}\right] = \frac{\mathrm{Im}[\tilde{\mu}(\omega)]}{|\tilde{\mu}(\omega)|^2} > 0 \,, \tag{1.190}$$

and therefore

$$\mathrm{Re}\left\{2\mathrm{i}\omega \left(\langle w_e(\vec{r}) \rangle - \langle w_m(\vec{r}) \rangle \right)\right\} = \frac{1}{2}\omega \left(\mathrm{Im}[\tilde{\epsilon}(\omega)] \, \left|\vec{E}(\vec{r})\right|^2 + \frac{\mathrm{Im}[\tilde{\mu}(\omega)]}{|\tilde{\mu}(\omega)|^2} \, \left|\vec{B}(\vec{r})\right|^2 \right) > 0 \,. \tag{1.191}$$

We recognize the single-frequency component of Eq. (1.171). We remember that $\langle P(\vec{r}) \rangle$ is the power density for the work done by the electric field on the constituent atoms in the sample, i.e., with the electric field, not against it. So, $\langle P(\vec{r}) \rangle$ is positive when the "battery" is discharged, negative when the "battery" is being charged. This power density is thus negative when electromagnetic radiation is being absorbed by the sample, or when the condition (1.191) is fulfilled.

The prefactors in Eq. (1.185) can be understood as time averages over one oscillation period of the continuous-wave field, as in the central result

$$\frac{1}{T}\int_0^{T=2\pi/\omega} \mathrm{d}t\, \cos^2(\omega t) = \frac{1}{2}\,, \tag{1.192}$$

where $\omega = 2\pi/T$ is the angular frequency, and T is the oscillation period.

1.3.4 Intermezzo: Maxwell Equations for a Plane Wave

Let us consider a situation without a manifest charge and current density, and $\tilde{\epsilon}(\omega) = \epsilon_0$ and $\tilde{\mu}(\omega) = \mu_0$. We now consider a plane wave,

$$\begin{aligned} \vec{E}(\vec{r},t) &= \vec{E}_0\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot r - \mathrm{i}\omega t}\,, \qquad \vec{B}(\vec{r},t) = \vec{B}_0\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot r - \mathrm{i}\omega t}\,, \\ \vec{E}(\vec{r}) &= \vec{E}_0\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot r}\,, \qquad \vec{B}(\vec{r}) = \vec{B}_0\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot r}\,, \qquad \rho(\vec{r},t) = 0\,, \qquad \vec{J}(\vec{r},t) = \vec{0}\,, \end{aligned} \tag{1.193}$$

then the Maxwell equations become ($\vec{\nabla} \to \mathrm{i}\,\vec{k}$)

$$\begin{aligned} \vec{k}\cdot\vec{E}_0 &= 0\,, \qquad \vec{k}\cdot\vec{B}_0 = 0\,, \\ \mathrm{i}\,\vec{k}\times\vec{E}_0 - \mathrm{i}\omega\vec{B}_0 &= 0\,, \qquad \mathrm{i}\,\vec{k}\times\vec{B}_0 + \mathrm{i}\frac{\omega}{c^2}\vec{E}_0 = \vec{0}\,. \end{aligned} \tag{1.194}$$

The magnetic field amplitude is given as

$$\vec{B}_0 = \frac{\vec{k}}{\omega}\times\vec{E}_0\,. \tag{1.195}$$

Then, the Ampere-Maxwell law is fulfilled,

$$\vec{k}\times\vec{B}_0 = \vec{k}\times\left(\frac{\vec{k}}{\omega}\times\vec{E}_0\right) = -\frac{\vec{k}^2}{\omega}\,\vec{E}_0 = -\frac{\omega}{c^2}\,\vec{E}_0\,, \tag{1.196}$$

where we have used the relation $\vec{k}^2 = \omega^2/c^2$. The Poynting vector is

$$\langle\vec{S}\rangle = \frac{1}{2\mu_0}\vec{E}_0\times\vec{B}_0 = \frac{1}{2\mu_0}\left(\vec{E}_0\times\left(\frac{\vec{k}}{\omega}\times\vec{E}_0^*\right)\right) = \frac{\vec{k}}{2\,\omega\,\mu_0}\,\left|\vec{E}_0\right|^2\,. \tag{1.197}$$

The energy densities evaluate as

$$\langle w_e\rangle = \frac{1}{4}\epsilon_0\,\left|\vec{E}_0\right|^2\,, \tag{1.198}$$

$$\langle w_m\rangle = \frac{1}{4}\frac{1}{\mu_0}\left|\vec{B}_0\right|^2 = \frac{1}{4\mu_0}\left|\frac{\vec{k}}{\omega}\times\vec{E}_0\right|^2 = \frac{1}{4\mu_0 c^2}\left|\vec{E}_0\right|^2 = \langle w_e\rangle\,. \tag{1.199}$$

This illustrates once more that the energy stored in the electric and magnetic fields describing a traveling wave, are equal.

Chapter 2

Green Functions for the Wave Equation

2.1 Orientation

This chapter is devoted to a discussion of Green functions. We discuss the Green function of the wave equation in Sec. 2.2, in the three variants of the retarded, the advanced, and the Feynman Green function for the inhomogeneous wave equation. Applications of the retarded Green function and the problem of the action-at-a-distance are discussed in Sec. 2.3. In Sec. 2.4, we devote special attention to the Green function of the damped, harmonic oscillator. We then apply the concept to the description of an atom, whose transitions can be modeled as damped harmonic oscillators, and briefly discuss wave propagation and phase shifts near the end of the current chapter.

2.2 Green Function for the Wave Equation

2.2.1 Derivation of the Basic Expression

The wave equation, with sources, has the general form

Wave Equation with Sources: $$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\psi\left(\vec{r},t\right) = \frac{1}{\epsilon_0}\,F\left(\vec{r},t\right)\,. \tag{2.1}$$

One approach to the solution of this equation is to work with the space-time Fourier transforms of the fields and sources. The space-time functions are related to the corresponding functions in Fourier space (which are functions of the wave vector and of the angular frequency) by

$$\psi\left(\vec{r},t\right) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\int\frac{\mathrm{d}\omega}{2\pi}\,\widetilde{\psi}(\vec{k},\omega)\,\exp\left(\mathrm{i}\left[\vec{k}\cdot\vec{r}-\omega t\right]\right)\,, \tag{2.2}$$

$$F\left(\vec{r},t\right) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\int\frac{\mathrm{d}\omega}{2\pi}\,\widetilde{F}(\vec{k},\omega)\,\exp\left(\mathrm{i}\left[\vec{k}\cdot\vec{r}-\omega t\right]\right)\,, \tag{2.3}$$

with the Fourier backtransformation

$$\widetilde{\psi}(\vec{k},\omega) = \int \psi(\vec{r},t)\ \exp\left(-\mathrm{i}\left[\vec{k}\cdot\vec{r}-\omega t\right]\right)\ \mathrm{d}^3r\,\mathrm{d}t\,, \tag{2.4}$$

$$\widetilde{F}(\vec{k},\omega) = \int F(\vec{r},t)\ \exp\left(-\mathrm{i}\left[\vec{k}\cdot\vec{r}-\omega t\right]\right)\ \mathrm{d}^3r\,\mathrm{d}t\,, \tag{2.5}$$

where we use the relationship

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}u\,v}\mathrm{d}v = \delta\left(u\right)\,. \tag{2.6}$$

We may consider the Fourier transform of Eq. (2.1),

$$\begin{aligned}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\int\frac{\mathrm{d}\omega}{2\pi}\ \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\vec{\nabla}^2\right)\psi(\vec{k},\omega)\exp\left(\mathrm{i}\left[\vec{k}\cdot\vec{r}-\omega t\right]\right)\\ = \frac{1}{\epsilon_0}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\int\frac{\mathrm{d}\omega}{2\pi}\ \widetilde{F}(\vec{k},\omega)\ \exp\left(\mathrm{i}\left[\vec{k}\cdot\vec{r}-\omega t\right]\right)\,.\end{aligned} \tag{2.7}$$

Because the differential operators only act on the exponential terms, the equation is converted into an algebraic equation,

$$\int \frac{\mathrm{d}^3k}{(2\pi)^3}\int\frac{\mathrm{d}\omega}{2\pi}\ \left[\psi(\vec{k},\omega)\left(-\vec{\nabla}^2+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)-\frac{1}{\epsilon_0}\widetilde{F}(\vec{k},\omega)\right]\exp\left(\mathrm{i}\left[\vec{k}\cdot\vec{r}-\omega t\right]\right)=0\,, \tag{2.8}$$

and so

$$\int \frac{\mathrm{d}^3k}{(2\pi)^3}\int\frac{\mathrm{d}\omega}{2\pi}\ \left[\psi(\vec{k},\omega)\left(\vec{k}\cdot\vec{k}-\frac{\omega^2}{c^2}\right)-\frac{1}{\epsilon_0}\widetilde{F}(\vec{k},\omega)\right]\exp\left(\mathrm{i}\left[\vec{k}\cdot\vec{r}-\omega t\right]\right)=0\,. \tag{2.9}$$

Since each Fourier component $\exp(\mathrm{i}\,[\vec{k}\cdot\vec{r}-\omega t])$ is linearly independent, all the Fourier coefficients must be zero. In other words, if a function vanishes, then its Fourier transform vanishes, and vice versa. The solution of the algebraic equation fulfilled by the Fourier transform is

Wave Equation in Fourier Space: $$\widetilde{\psi}(\vec{k},\omega) = \frac{1}{\epsilon_0}\frac{\widetilde{F}(\vec{k},\omega)}{\vec{k}^2-(\omega/c)^2}\,. \tag{2.10}$$

The problem is now reduced to taking the inverse Fourier transform. The most immediate approach is to integrate

$$\begin{aligned}\psi(\vec{r},t) &= \int\frac{\mathrm{d}^3k}{(2\pi)^3}\int\frac{\mathrm{d}\omega}{2\pi}\ \widetilde{\psi}(\vec{k},\omega)\ \exp\left(\mathrm{i}\left[\vec{k}\cdot\vec{r}-\omega t\right]\right)\\ &= \int\frac{\mathrm{d}^3k}{(2\pi)^3}\int\frac{\mathrm{d}\omega}{2\pi}\ \frac{1}{\epsilon_0}\ \frac{\exp\left(\mathrm{i}\left[\vec{k}\cdot\vec{r}-\omega t\right]\right)}{\vec{k}^2-(\omega/c)^2}\ \widetilde{F}(\vec{k},\omega)\\ &= \int\mathrm{d}^3r'\int\mathrm{d}t'\int\frac{\mathrm{d}^3k}{(2\pi)^3}\int\frac{\mathrm{d}\omega}{2\pi}\ \frac{1}{\epsilon_0}\ \frac{\exp\left(\mathrm{i}\left[\vec{k}\cdot(\vec{r}-\vec{r}')-\omega(t-t')\right]\right)}{\vec{k}^2-(\omega/c)^2}\ F(\vec{r}',t')\\ &= \int\mathrm{d}^3r'\int\mathrm{d}t'\ \underbrace{\frac{1}{\epsilon_0}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\int\frac{\mathrm{d}\omega}{2\pi}\ \frac{\exp\left(\mathrm{i}\left[\vec{k}\cdot(\vec{r}-\vec{r}')-\omega(t-t')\right]\right)}{\vec{k}^2-(\omega/c)^2}}_{=G(\vec{r}-\vec{r}',t-t')}\ F(\vec{r}',t')\\ &= \int\mathrm{d}^3r'\int\mathrm{d}t'\,G(\vec{r}-\vec{r}',t-t')\,F(\vec{r}',t')\end{aligned} \tag{2.11}$$

where we have used the concept that a Green function connects the source $F(\vec{r}',t')$ to the response $\psi(\vec{r},t)$.

Another approach is based on the observation that the Green function $G(\vec{r}-\vec{r}', t-t')$ with its Fourier transform $G(\vec{k},\omega)$ will solve Eq. (2.1) with the source term given by

$$F(\vec{r},t) = \delta^{(3)}(\vec{r}-\vec{r}')\,\delta(t-t') = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{\mathrm{d}\omega}{2\pi}\,\exp\left(\mathrm{i}\left[\vec{k}\cdot(\vec{r}-\vec{r}') - \omega\,(t-t')\right]\right)\,, \tag{2.12}$$

i.e.

$$\boxed{\text{Green Function in Coordinate Space:}} \quad \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right) G\,(\vec{r}-\vec{r}', t-t') = \frac{1}{\epsilon_0}\,\delta^{(3)}(\vec{r}-\vec{r}')\,\delta\,(t-t')\,. \tag{2.13}$$

There is a certain problem in how to denote the dependence on $\vec{r}'$ in the Green function. We here adopt the convention that

$$G\,(\vec{r},t,\vec{r}',t') = G\,(\vec{r}-\vec{r}',t-t')\,, \tag{2.14}$$

while the differential operators only act on the $\vec{r}$ coordinate. One can find the Fourier transform $\widetilde{G}(\vec{k},\omega)$ of the Green function by transforming Eq. (2.13) into Fourier space, i.e., by applying the differential operators under the integral sign, to the Fourier transform,

$$\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{\mathrm{d}\omega}{2\pi}\left[\widetilde{G}(\vec{k},\omega)\left(-\vec{\nabla}^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) - \frac{1}{\epsilon_0}\right]\exp\left(\mathrm{i}\left[\vec{k}\cdot(\vec{r}-\vec{r}') - \omega(t-t')\right]\right) = 0\,, \tag{2.15}$$

or

$$\int \left[\widetilde{G}(\vec{k},\omega)\left(\vec{k}\cdot\vec{k} - \frac{\omega^2}{c^2}\right) - \frac{1}{\epsilon_0}\right]\exp\left(\mathrm{i}\left[\vec{k}\cdot(\vec{r}-\vec{r}') - \omega(t-t')\right]\right) = 0\,. \tag{2.16}$$

In Fourier space, thus,

$$\boxed{\text{Green Function in Fourier Space:}} \qquad \widetilde{G}(\vec{k},\omega) = \frac{1}{\epsilon_0}\,\frac{1}{\vec{k}^2 - (\omega/c)^2}\,. \tag{2.17}$$

Furthermore, in terms of space-time coordinates, the Green function for the wave equation is found by Fourier backtransformation

$$G\,(\vec{r}-\vec{r}',t-t') = \frac{1}{\epsilon_0}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\int\frac{\mathrm{d}\omega}{2\pi}\,\frac{\exp\left(\mathrm{i}\left[\vec{k}\cdot(\vec{r}-\vec{r}') - \omega\,(t-t')\right]\right)}{\vec{k}^2 - (\omega/c)^2}\,, \tag{2.18}$$

as already anticipated in Eq. (2.11). The integrand has singularities at $\omega = \pm c\,k$ (or $k = \pm\omega/c$). How these singularities are handled depends on the boundary conditions imposed on the system. Only after the "$\mathrm{i}\epsilon$" prescription has been implemented is the Green function uniquely defined. The basic approach involves *Cauchy's Residue Theorem.*

Of special importance in evaluating Eq. (2.18) is the ω integral. In view of the identity

$$\frac{1}{\vec{k}^2 - \omega^2/c^2} = \frac{c^2}{c^2\,\vec{k}^2 - \omega^2} = \frac{c}{2\,|\vec{k}|}\left(\frac{\omega - c\,|\vec{k}|}{\omega^2 - (c\,|\vec{k}|)^2} - \frac{\omega + c\,|\vec{k}|}{\omega^2 - (c\,|\vec{k}|)^2}\right) = \frac{c}{2\,|\vec{k}|}\left(\frac{1}{\omega + c\,|\vec{k}|} - \frac{1}{\omega - c\,|\vec{k}|}\right)\,, \tag{2.19}$$

we can isolate the $\mathrm{d}\omega$ integral as follows,

$$\begin{aligned} G\,(\vec{r}-\vec{r}',t-t') &= \int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{c}{2|\vec{k}|\,\epsilon_0}\,\exp\left(\mathrm{i}\,\vec{k}\cdot(\vec{r}-\vec{r}')\right) \\ &\quad\times\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\,\omega\,(t-t')}\left(\frac{1}{\omega + c\,|\vec{k}|} - \frac{1}{\omega - c\,|\vec{k}|}\right)\,. \end{aligned} \tag{2.20}$$

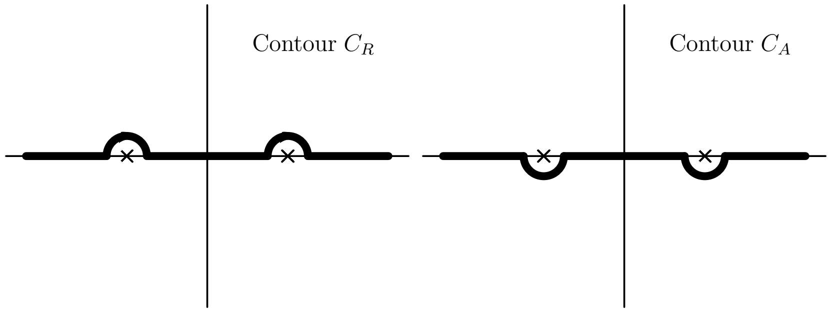


Figure 2.1: Location and direction of the contours C_R and C_A relative to the real ω axis (center line). For $t > t'$, the contour C_R used in evaluating the integral in Eq. (2.23) is completed below the real axis and encircles the poles. For $t < t'$, we have to close the contour in the upper complex half plane, and the result for the integral (2.23) is zero. For $t < t'$, the contour C_A is completed above the real axis so that the poles are encircled. For $t > t'$, the contour C_A is completed below the real axis and P_C vanishes.

The integration interval $(-\infty, +\infty)$ of the ω integration has to be suitably continued into the complex plane in order for the Cauchy residue theorem to be applicable, which implies that we have to encircle the poles at $\omega = \pm|\vec{k}|$ either infinitesimally above, or infinitesimally below the real axis. Indeed, the integrand has two first order poles along the real ω axis, which leads to several possible choices of the integration contour $(-\infty, \infty) \to C$. We write G as follows,

$$G\,(\vec{r}-\vec{r}', t-t') = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\; f(\vec{k})\; P_C(t-t', |\vec{k}|) \tag{2.21}$$

where

$$f(\vec{k}) = \frac{c}{2|\vec{k}|\epsilon_0}\; \exp\left(\mathrm{i}\,\vec{k}\cdot(\vec{r}-\vec{r}')\right) \tag{2.22}$$

and

$$\boxed{\textbf{Frequency Integral:}} \qquad P_C(t-t', |\vec{k}|) = \oint\limits_{\text{closed}} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\,\omega\,(t-t')} \left(\frac{1}{\omega + c\,|\vec{k}|} - \frac{1}{\omega - c\,|\vec{k}|} \right) \tag{2.23}$$

depends on the choice of the contour C.

In principle, as already explained, we have several possibilities for distorting the path, three of which are being considered here ($C = C_R$, $C = C_A$, and $C = C_F$). The pertinent paths are shown in Figs. 2.1 and 2.2. The figure captions provide certain explanations which will be supplemented in the following. Writing $k = |\vec{k}|$, we recall that the ω or angular frequency part of the integrand reads as follows,

$$P_C(t-t', k) = \int\limits_C \frac{\mathrm{d}\omega}{2\pi} \exp(-\mathrm{i}\,\omega\,(t-t')) \left(\frac{1}{\omega + c\,k} - \frac{1}{\omega - c\,k} \right), \tag{2.24}$$

where C is the path of integration. We will always close the integration path C through a semi-circle, with infinite radius, in either the upper half complex ω plane or in the lower complex ω half plane. The denominator will cause the integrand to vanish, on either semi-circle, as $|\omega|^{-2}$ for large $|\omega|$ if the numerator is well behaved. It thus suffices to check the numerator on the imaginary ω axis, i.e., for $\omega = \mathrm{i}\,\mathrm{Im}(\omega)$. The

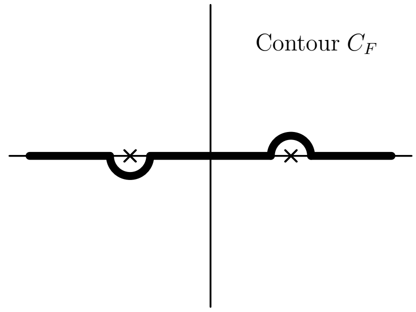


Figure 2.2: Feynman contour C_F corresponding to the time-ordered product of photon field operators in field theory.

value is determined by the exponential

$$\left|\exp\left[-\mathrm{i}\,\omega\,(t-t')\right]\right| = \exp\left[\mathrm{Im}(\omega)\ (t-t')\right]. \tag{2.25}$$

If $t - t' > 0$ and $\mathrm{Im}(\omega) \to +\infty$, then this expression diverges. However, if $t - t' > 0$ and $\mathrm{Im}(\omega) \to -\infty$, then this expression vanishes exponentially. Thus, for $t > t'$ the path used in evaluating (2.23) is closed in the lower half of the complex ω plane, which means that the poles are being encircled from above, and this gives a nonvanishing contribution for the retarded Green function (contour C_R). Conversely, for $t < t'$, we close the path along a semi-circle in the upper half of the complex ω plane, encircle the poles from below, and this integration yields a nonvanishing contribution for the advanced Green function (contour C_A).

Let us discuss the closure of the integration contour on the basis of a different argument. The contour $C \to [C + \text{semi-circle}]$ gets closed either in the upper or in the lower half of the complex plane. There is thus a compensating term which one has to subtract, as the contribution of the closing semi-circle is added to the interval $(-\infty, +\infty)$ in order to form the closed contour. In our derivations, we assume that the contribution of the semi-circle to the integral,

$$Q_C(t-t',k) = -\int\limits_{\text{semi-circle}} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\,\omega(t-t')} \left(\frac{1}{\omega + c\,|\vec{k}|} - \frac{1}{\omega - c\,|\vec{k}|}\right) \tag{2.26}$$

vanishes because of the above mentioned arguments. Unmarked exercise: Verify these statements based on your own independent calculations. In particular, choose for ω along the semi-circle a representation of the form $R\,\exp(\mathrm{i}\,\theta)$ with $0 \le \theta < \pi$, or $-\pi < \theta \le 0$, based on a closure in the upper or lower complex plane. In the following, we will thus assume that all contour integrals defining P_C along the integration paths C are closed in either the upper or lower complex plane, leading to a convergent integral which is defined by a closed contour $[C + \text{semi-circle}]$. This contour can be "shrunk" by virtue of Cauchy's residue theorem, to a contour which encloses only the poles.

Because the poles in the frequency integral (2.23) lie on the real axis, the integral actually is not well-defined, leading to different Green functions obtained by integration along the different contours C_R, C_A and C_F. However, we should clarify that all integration paths lead to valid solutions of the defining equation (2.13), i.e., they lead to solutions of

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right) G\left(\vec{r} - \vec{r}', t-t'\right) = \frac{1}{\epsilon_0}\,\delta^{(3)}\left(\vec{r} - \vec{r}'\right)\,\delta\left(t-t'\right). \tag{2.27}$$

In order to see this, one applies the differential operator under the integral sign, in the Fourier representation of the integral. This cancels the problematic denominator; the contours C_R, C_A and C_F can thus be redeformed into the integrations along the real axis, from $(-\infty, \infty)$, and the integral can be evaluated in terms of a Dirac-δ function. This constitutes a second, unmarked exercise given in this, current section. The different solutions just fulfill different boundary conditions, as is natural for second-order partial differential equations. Indeed, the ambiguity provides the freedom to impose temporal boundary conditions on the Green function. As much as we saw in electrostatics that we can impose different boundary conditions on well-defined two-dimensional manifolds imbedded in the three-dimensional space (so-called Dirichlet and Neumann boundary conditions), we can now impose boundary conditions on three-dimensional manifolds imbedded in the four-dimensional space-time continuum. The standard Green function of electrostatics, being proportional to $1/|\vec{r} - \vec{r}'|$, vanishes on the surface of a large sphere with infinite radius, which is a two-dimensional manifold. By contrast, the above Green function vanishes for large distances and/or for large time differences, i.e., on a three-dimensional manifold.

2.2.2 Connection to Electrostatics

Our basic, paradigmatic integral in graduate electrodynamics I was given by the formula

$$\Phi(\vec{r}) = -\frac{1}{\epsilon_0} \int \mathrm{d}^3 r' \, g(\vec{r} - \vec{r}') \rho(\vec{r}') \,, \tag{2.28}$$

where Φ is the electrostatic, scalar potential, ρ is the charge density, and

$$g(\vec{r} - \vec{r}') = -\frac{1}{4\pi |\vec{r} - \vec{r}'|} \,, \qquad \vec{\nabla}^2 g(\vec{r} - \vec{r}') = \delta^{(3)}(\vec{r} - \vec{r}') \tag{2.29}$$

is the Green function of electrostatics (with boundary conditions set for Φ to vanish at infinity).

The connection to electrodynamics can be formulated as follows. We first observe that electrostatics investigates static field configurations, which in turn corresponds to slowly changing (constant in time) fields, i.e., to an interrogation at zero frequency. We thus investigate a mixed representation of Eq. (2.18),

$$G\left(\vec{r} - \vec{r}', t - t'\right) = \frac{1}{\epsilon_0} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \int \frac{\mathrm{d}\omega}{2\pi} \, \frac{\exp\left(\mathrm{i}\left[\vec{k} \cdot (\vec{r} - \vec{r}') - \omega\,(t - t')\right]\right)}{\vec{k}^2 - (\omega/c)^2} \,, \tag{2.30}$$

which reads as

$$G\left(\vec{r} - \vec{r}', \omega\right) = \frac{1}{\epsilon_0} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \, \frac{\exp\left[\mathrm{i}\,\vec{k} \cdot (\vec{r} - \vec{r}')\right]}{\vec{k}^2 - (\omega/c)^2} \,. \tag{2.31}$$

In the static limit $\omega \to 0$, it follows that

$$G\left(\vec{r} - \vec{r}', \omega \to 0\right) = \frac{1}{\epsilon_0} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \int \frac{\mathrm{d}\omega}{2\pi} \, \frac{\mathrm{e}^{\mathrm{i}\,\vec{k}\cdot(\vec{r}-\vec{r}')}}{\vec{k}^2} = \frac{1}{4\pi\,\epsilon_0\,|\vec{r} - \vec{r}'|} \,. \tag{2.32}$$

In turn, in the limit $\omega \to 0$, the defining equation for the Green function of the wave equation becomes

$$\begin{aligned} \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right) G\left(\vec{r} - \vec{r}', t - t'\right) &= \frac{1}{\epsilon_0}\, \delta^{(3)}\left(\vec{r} - \vec{r}'\right)\, \delta\left(t - t'\right) \\ \Rightarrow \qquad \left(-\vec{\nabla}^2\right) G\left(\vec{r} - \vec{r}', \omega \to 0\right) &= \frac{1}{\epsilon_0}\, \delta^{(3)}\left(\vec{r} - \vec{r}'\right) \,. \end{aligned} \tag{2.33}$$

The latter equation, in turn, is consistent with Eq. (2.32) and also consistent with Eq. (2.29).

We thus have the identification

$$g(\vec{r}-\vec{r}') = -\frac{1}{4\pi|\vec{r}-\vec{r}'|}, \qquad \vec{\nabla}^2 g(\vec{r}-\vec{r}') = \delta^{(3)}(\vec{r}-\vec{r}'), \tag{2.34a}$$

$$G(\vec{r}-\vec{r}', \omega \to 0) = \frac{1}{4\pi\,\epsilon_0\,|\vec{r}-\vec{r}'|}, \qquad \vec{\nabla}^2 G(\vec{r}-\vec{r}', \omega \to 0) = -\frac{1}{\epsilon_0}\,\delta^{(3)}(\vec{r}-\vec{r}') \tag{2.34b}$$

$$G(\vec{r}-\vec{r}', \omega \to 0) = -\frac{1}{\epsilon_0}\,g(\vec{r}-\vec{r}'). \tag{2.34c}$$

The integrand in Eq. (2.30) has singularities at $\omega = \pm c\left|\vec{k}\right|$, which need to be handled when doing the ω integral. The different choices of contour will later be understood as the different choices for the boundary conditions imposed on the Green function. In the case of electrostatics, we saw that we could construct different Green functions fulfilling the same equation

$$\vec{\nabla}^2 g(\vec{r}-\vec{r}') = \delta^3(\vec{r}-\vec{r}'), \tag{2.35}$$

but with different boundary conditions, by adding to the Green function a solution of the homogeneous equation. Likewise, here, we will see that the Green function can fulfill its defining equation and still, fulfill different boundary conditions in space and time (here, including time, because the defining equation for the Green function lives in four-dimensional space-time). Related questions will be handled in the following.

2.2.3 Cauchy's Residue Theorem: A Small Digression

The Taylor expansion of a function $f = f(x)$ about the origin reads as

$$f(x) = f(0) + f'(0)\,x + \frac{1}{2!}f''(0)\,x^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\,x^n = \sum_{n=0}^{\infty} a_n\,x^n. \tag{2.36}$$

Here, the a_n with $n = 0, 1, 2, \ldots$ are constant coefficients. However, functions like

$$g(z) = \frac{1}{z^4 \sin(z)} \tag{2.37}$$

obviously cannot be expanded in a Taylor series about $z = 0$. At best, we can do the following expansion,

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots, \tag{2.38}$$

and so

$$\begin{aligned} g(z) = \frac{1}{z^4 \sin(z)} &= \frac{1}{z^4\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots\right)} = \frac{1}{z^5\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \ldots\right)} \\ &= \frac{1}{z^5} + \frac{1}{6\,z^3} + \frac{7}{360\,z} + \frac{31\,z}{15120} + \mathcal{O}(z^3). \end{aligned} \tag{2.39}$$

This expansion obviously does not start from the term of order zero, but from a term proportional to z^{-5}. It is called a Laurent expansion. The general formula reads

$$\boxed{\text{Laurent expansion:}} \qquad g(z) = \sum_{n=-\infty}^{\infty} a_n\,z^n. \tag{2.40}$$

It starts at $n = -\infty$ instead of $n = 0$, like the Taylor expansion. Of course, we can formulate a Laurent expansion about a point z_0, which reads

$$g(z) = \sum_{n=-\infty}^{\infty} b_n \, (z - z_0)^n \tag{2.41}$$

with different coefficients b_n, but let us stick for the moment with the expansion (2.40). (If all coefficient b_n with $n < 0$ vanish, we say that $g(z)$ is analytic about the point $z = z_0$. If all a_n's with $n < -M$ will vanish, then we say that g has an Mth order pole at the origin.) Of preeminent importance is the integral

$$I = \oint g(z) \, \mathrm{d}z = \int_C g(z) \, \mathrm{d}z = \int_C \left(\sum_{n=-\infty}^{\infty} a_n \, z^n \right) \mathrm{d}z \,, \tag{2.42}$$

where the contour C is an anticlockwise circle (taken in the mathematically positive sense) of radius R around the origin. We shall see that only a very limited number of a_n coeffcients (in fact, just one of them) contributes to the integral. Let us therefore consider the integral

$$J = \oint_C \mathrm{d}z \, \frac{1}{z^m} \,, \qquad z = R \, \exp(\mathrm{i}\theta) \,, \qquad \theta \in (0, 2\pi) \,. \tag{2.43}$$

where $z = z(\theta)$ thus describes an anticlockwise circle about the origin. Furthermore, m will be assumed to be an integer. Now, for $m \neq 1$, we have

$$\begin{aligned} J &= \int_0^{2\pi} \mathrm{d}\theta \; (\mathrm{i}R) \; \exp(\mathrm{i}\theta) \; [R \exp(\mathrm{i}\theta)]^{-m} = \mathrm{i}R^{1-m} \int_0^{2\pi} \mathrm{d}\theta \; \exp[-\mathrm{i}(m-1)\theta] \\ &= \mathrm{i}R^{1-m} \left[\frac{1}{-(m-1)\mathrm{i}} \; \exp[-\mathrm{i}(m-1)\theta] \right]_{\theta=0}^{\theta=2\pi} = -R^{1-m} \left[\frac{1}{(m-1)} \; \exp[-\mathrm{i}(m-1)\theta] \right]_{\theta=0}^{\theta=2\pi} = 0 \,, \quad m \neq 1 \,. \end{aligned} \tag{2.44}$$

However, for $m = 1$, we can replace $m - 1 \to \epsilon$ and do a Taylor expansion,

$$\begin{aligned} \lim_{m \to 1} J = &\; - R^{-\epsilon} \left[\lim_{\epsilon \to 0} \frac{1}{\epsilon} \; \exp(-\mathrm{i}\epsilon\theta) \right]_{\theta=0}^{\theta=2\pi} = - \left[\frac{1}{\epsilon} \; (1 - \mathrm{i}\epsilon\theta) \right]_{\theta=0}^{\theta=2\pi} \\ = &\; - \left[\frac{1}{\epsilon} - \mathrm{i}\,\theta \right]_{\theta=0}^{\theta=2\pi} = [\mathrm{i}\,\theta]_{\theta=0}^{\theta=2\pi} = 2\pi\,\mathrm{i} \,. \end{aligned} \tag{2.45}$$

Of course, this result could have been obtained by a direct calculation, without letting $m \to 1$, but the above derivation illustrates that the special case $m = 1$ actually follows from the more general case by a limiting process. For a function $g(z)$ that has a Laurent expansion of the form $g(z) = \sum_{n=-\infty}^{\infty} a_n \, z^n$, we therefore have

$$I = \oint_C \mathrm{d}z \, f(z) = \oint_C \mathrm{d}z \left(\sum_{m=-\infty}^{\infty} a_m \, z^m \right) = 2\pi\,\mathrm{i}\; a_{-1} \equiv 2\pi \; \operatorname*{Res}_{z=0} \; f(z) \,, \tag{2.46}$$

where the last term constitutes a definition of the residue. Only the coefficient a_{-1} contributes. We write this as

$$\operatorname*{Res}_{z=0} \; f(z) \equiv a_{-1} \,, \qquad \oint_C \mathrm{d}z \, f(z) = 2\pi\,\mathrm{i} \; \operatorname*{Res}_{z=0} \; f(z) \,. \tag{2.47}$$

Since closed contours in the complex z plane do not normally contribute any nonvanishing value to an integral, this result can be generalized easily, as follows. Namely, for an expansion about an arbitrary point z_0, we have

$$f(z) = \sum_{m=-\infty}^{\infty} b_m \, (z - z_0)^m \tag{2.48}$$

and a contour integral that encircles z_0 in the mathematically positive sense,

$$\oint_{C_0} \mathrm{d}z\, f(z) = 2\pi\,\mathrm{i}\; b_{-1} = 2\pi\,\mathrm{i}\; \operatorname*{Res}_{z=z_0}\; f(z)\,. \tag{2.49}$$

Closed contour integrals in regions where the integrand is analytic, simply vanish. This enables us to generalize the result as follows. For a contour $\mathcal{C}$ that encircles n residues located at point $z = z_i$ with $i \in \{1, \dots, n\}$, the result is as follows,

$$\oint_{\mathcal{C}} \mathrm{d}z\, f(z) = 2\pi\,\mathrm{i}\; \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i}\; f(z)\,. \tag{2.50}$$

This is Cauchy's residue theorem. For functions that have a single pole, like

$$g(z) = \frac{1}{\sin z} \approx \frac{1}{z - \frac{1}{3}\, z^3 + \dots}\,, \qquad \operatorname*{Res}_{z=0}\; g(z) = 1\,, \tag{2.51}$$

the calculation of the residue is obvious. For functions that have poles of higher order, it becomes more complicated. We have already derived that

$$\operatorname*{Res}_{z=0} \left(\frac{1}{z^4\; \sin(z)} \right) = \frac{7}{360}\,. \tag{2.52}$$

Let us also consider

$$g(z) = \frac{\cos(z)}{z^4\; \sin(z)} = \frac{1}{z^5} \frac{z\cos(z)}{\sin(z)}\,. \tag{2.53}$$

Now, a simple Taylor expansion of numerator and denominator shows that

$$\frac{z\cos(z)}{\sin(z)} = 1 - \frac{z^2}{3} - \frac{z^4}{45} + \dots \tag{2.54}$$

and so, for $z \to 0$,

$$g(z) \approx \frac{1}{z^5} \left(1 - \frac{z^2}{3} - \frac{z^4}{45} + \dots \right)\,, \qquad \operatorname*{Res}_{z=0}\; g(z) = -\frac{1}{45}\,. \tag{2.55}$$

Let us now investigate a generalization of this result, namely, the case where $f(z)$ has an Mth order pole. For the example in Eq. (2.53), we have $M = 5$. We are interested in finding out about the term of order $(z - z_0)^{-1}$ in $f(z)$. To this end, one can follow the *ad hoc* procedure outlined above: We just expand numerator and denominator to the required order and then read off the term of order $(z - z_0)^{-1}$. However, it is also possible to follow a more systematic approach. If one multiplies the terms $\{a_{-M}\,(z - z_0)^{-M}, a_{-M+1}\,(z - z_0)^{-M+1}, \dots, a_{-1}\,(z - z_0)^{-1}, \dots\}$ by a factor $(z - z_0)^M$, then they transform into the set $\{a_{-M}, a_{-M+1}\,(z - z_0), \dots, a_{-1}\,(z - z_0)^{M-1}, \dots\}$. Expressed differently,

$$f(z) = \frac{a_{-M}}{(z - z_0)^M} + \frac{a_{-M+1}}{(z - z_0)^{M-1}} + \dots \frac{a_{-1}}{(z - z_0)} + \dots + a_n\, z^n + \dots\,, \tag{2.56}$$

$$(z - z_0)^M\, f(z) = a_{-M} + (z - z_0)\, a_{-M+1} + \dots + (z - z_0)^{M-1}\, a_{-1} + \dots\,. \tag{2.57}$$

If we differentiate the expression $(z - z_0)^M\, f(z)$ a total of $M - 1$ times with respect to z, and set $z = z_0$ after the differentiation, then the only contributing term is the one proportional to a_{-1}. Terms of lower order are explicitly annihilated by the differential operator, and terms of higher order give rise to positive powers of $(z - z_0)$; these are annihilated if we set $z = z_0$ after the differentiation. This leads to the general formula, valid for an Mth order pole,

$$\operatorname*{Res}_{z=z_0}\; f(z) = \frac{1}{(M-1)!} \left[\frac{\mathrm{d}^{M-1} \left\{ (z - z_0)^M\;\; f(z) \right\}}{\mathrm{d}z^{M-1}} \right]_{z=z_0}\,. \tag{2.58}$$

Another application is as follows. Let us consider an integral such as

$$\int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\eta}{\omega^2 + \eta^2} = \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\eta}{(\omega + \mathrm{i}\,\eta)\,(\omega - \mathrm{i}\,\eta)} \,. \tag{2.59}$$

We can close the integration contour either in the upper or the lower complex plane because the half-circle integral behaves as $R^{-2+1} \to 0$ for $R \to \infty$. At the singularity, at $\omega = \mathrm{i}\eta$, we have

$$\frac{\eta}{(\omega + \mathrm{i}\,\eta)\,(\omega - \mathrm{i}\,\eta)} \approx \frac{\eta}{2\,\mathrm{i}\,\eta\,(\omega - \mathrm{i}\,\eta)} \tag{2.60}$$

and hence

$$\operatorname*{Res}_{\omega=\mathrm{i}\eta} \frac{\eta}{(\omega + \mathrm{i}\,\eta)\,(\omega - \mathrm{i}\,\eta)} = \frac{\eta}{2\,\mathrm{i}\,\eta} = \frac{1}{2\,\mathrm{i}} \,. \tag{2.61}$$

The result is

$$\int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\eta}{\omega^2 + \eta^2} = 2\pi\,\mathrm{i}\,\frac{1}{2\,\mathrm{i}} = \pi \,. \tag{2.62}$$

Reversing the argument, for closing the contour in the lower half of the complex plane, we have

$$\int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\eta}{\omega^2 + \eta^2} = (-2\pi\,\mathrm{i}) \left(-\frac{1}{2\,\mathrm{i}} \right) = \pi \,. \tag{2.63}$$

The first minus sign is due to the reversed sense of revolution around the singularity. The second minus sign is due to a sign change in the residue at $\omega = -\mathrm{i}\eta$. This is left as an exercise to the reader.

2.2.4 Why So Many Green Functions?

In the following considerations, we shall encounter the retarded, advanced, and Feynman Green functions. All of these solve the basic, defining equation for the Green function, given in Eq. (2.13), which we recall for convenience,

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \right) G\left(\vec{r} - \vec{r}', t - t'\right) = \frac{1}{\epsilon_0}\, \delta^{(3)}\left(\vec{r} - \vec{r}'\right)\, \delta\left(t - t'\right) \,. \tag{2.64}$$

However, the Green functions differ in their causal behaviour and in their analytic structure.

In terms of a classical interpretation, let us first consider the case where the Green function describes a string. The retarded Green function $G_R(x, t, x', t')$ gives the response of the string (initially at rest) to a unit of momentum applied to the string at a point in time t' at a point x' along the string, with the response being recorded at space-time point (x, t) with $t > t'$. The advanced Green function $G_A(x, t, x', t')$ gives the initial motion of the string such that a unit of momentum applied at (x', t') causes it to come to rest, where the initial configuration is described at the space-time points (x, t) with $t < t'$. The advanced Green function "propagates into the past", the retarded Green function "propagates into the future".

In electrodynamics, the retarded Green function describes outgoing waves [scalar and vector potentials $\Phi(\vec{r}, t)$ and $\vec{A}(\vec{r}, t)$], generated by the sources $\rho(\vec{r}', t')$ and $\vec{J}(\vec{r}', t')$, with $t > t'$, i.e., the electromagnetic potentials and fields are generated by a unit disturbance at $(\vec{r}', t')$, with the fields being zero for $t < t'$. The advanced Green function in electrodynamics describes incoming waves, i.e., the scalar and vector potentials $\Phi(\vec{r}, t)$ and $\vec{A}(\vec{r}, t)$] which converge to a unit disturbance at $(\vec{r}', t')$, with $t < t'$, with the fields being zero for $t > t'$.

The retarded and advanced Green functions fulfill boundary conditions which are primarily defined on a three-dimensional submanifold of four-dimensional space-time, i.e., the manifold with $t - t' = 0$. By virtue of continuity, one then has to fulfill this boundary condition in the entire half-space $t > t'$ or $t < t'$.

The Feynman Green function was originally devised by Stueckelberg and Feynman (actually, it would be Count E. C. G. von Stückelberg who wrote scientific articles in English, French and German, and Richard P. Feynman, one of the most admired and influential physicists of the 20th century). The Feynman Green function relies essentially on a complex formalism (Fourier transform of the source configuration). Positive-frequency components of the source configuration are interpreted as sources for outgoing waves, which propagate into the future, whereas negative-frequency components are interpreted as sinks of incoming waves, which propagate into the past. This enables one to describe both field quanta creation as well as field quanta annihilation processes with one and the same Green function.

The necessity for introducing the Feyman Green function comes from field theory, notably, quantum electrodynamics. Roughly speaking, in field theory, in order to describe processes which involve the time evolution of the quantum fields, one needs to consider the so-called time-ordered T product of field operators,

$$\langle 0 \,|\mathrm{T}\, A^\mu(x)\, A^\nu(x')|\, 0\rangle = \Theta(t-t')\, \langle 0\,| A^\mu(x)\, A^\nu(x')|\, 0\rangle + \Theta(t'-t)\, \langle 0\,| A^\mu(x')\, A^\nu(x)|\, 0\rangle \,, \tag{2.65}$$

where Θ is the Heaviside step function. Here, $x = (t, \vec r)$ is a space-time coordinate four-vector. The Green function

$$G_F^{\mu\nu}(x-x') = g^{\mu\nu}\, G_F(x-x') = \langle 0\,|\mathrm{T}\, A^\mu(x)\, A^\nu(x')|\, 0\rangle \tag{2.66}$$

contains both retarded $[\propto \Theta(t-t')]$ as well as advanced $[\propto \Theta(t'-t)]$ components and is proportional to the Feynman propagator G_F. The space-time metric is $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$.

A final point: The three Green functions corresponding to the integration contours C_R, C_A and C_F (see Figs. 2.1 and 2.2) differ by a solution of the homogeneous equation, i.e.,

$$\left(\frac{1}{c^2}\,\frac{\partial^2}{\partial t^2} - \vec\nabla^2\right)\, (G_F(\vec r - \vec r', t-t') - G_R(\vec r - \vec r', t-t')) = 0\,, \tag{2.67}$$

and the same relation holds for $G_F - G_A$, and $G_R - G_A$. In our electrostatic analogy the different Green functions would correspond to different solutions of the defining equation of the electrostatic Green function,

$$\vec\nabla^2 g(\vec r - \vec r') = \delta^{(3)}(\vec r - \vec r')\,, \tag{2.68}$$

with different boundary conditions, implemented for the "boundaries" of space-time which can include the half-planes $t > t'$ and $t' < t$ (see Sec. 2.2.2). Pursuing the analogy further, one may remark that the Green function of electrostatics has the representation $-1/\vec k^2$ in wave number space, but in coordinate space, one may add a solution of the homogeneous equation $\vec\nabla^2 g(\vec r - \vec r') = 0$, which leads to the Dirichlet and Neumann Green functions. Analogous considerations are true for the Green function of electrodynamics, with $-1/\vec k^2$ being replaced by $(\epsilon_0\, (\vec k^{\,2} - (\omega/c)^2))^{-1}$ [see Eq. (2.17)].

2.2.5 Retarded Green Function

If the path C_R for the retarded Green function in Fig. 2.1 is used and $t < t'$, then we may close the integration path in the upper half of the complex plane, and the result for G is zero. However, if C_R is used and $t > t'$, then the poles are encircled in the clockwise (mathematically negative) direction, and we obtain

$$\begin{aligned}
P_{C_R}(t-t', k) &= -\frac{2\pi\mathrm{i}}{2\pi}\,\Theta\,(t-t')\,\left[\operatorname*{Res}_{\omega=-ck}\left(\frac{\mathrm{e}^{-\mathrm{i}\,\omega\left(t-t'\right)}}{\omega + c\,|\vec k|}\right) - \operatorname*{Res}_{\omega=+ck}\left(\frac{\mathrm{e}^{-\mathrm{i}\,\omega\left(t-t'\right)}}{\omega - c\,|\vec k|}\right)\right] \\
&= -\mathrm{i}\,\Theta\,(t-t')\,\left\{\mathrm{e}^{-\mathrm{i}(-ck)(t-t')} - \mathrm{e}^{-\mathrm{i}(+ck)(t-t')}\right\} \\
&= -\mathrm{i}\,\Theta\,(t-t')\,\left\{\mathrm{e}^{\mathrm{i}ck(t-t')} - \mathrm{e}^{-\mathrm{i}ck(t-t')}\right\} \\
&= -\mathrm{i}\,\Theta\,(t-t')\,\{\cos[ck(t-t')] + \mathrm{i}\,\sin[ck(t-t')] - \cos[ck(t-t')] + \mathrm{i}\,\sin[ck(t-t')]\} \\
&= 2\,\Theta\,(t-t')\,\sin(c\,k\,(t-t')).
\end{aligned} \tag{2.69}$$

We recall that the decisive consideration for the calculation of the intgral $P_{C_R}\left(t-t',|\vec{k}|\right)$ is as follows: If $t>t'$, then $t-t'>0$, and we have to close the path in the lower half of the complex plane. Then,

$$\exp(-\mathrm{i}\,\omega\,(t-t'))\to 0\,,\qquad \mathrm{Im}(\omega)<0\,,\qquad |\omega|\to\infty\,,\qquad t-t'>0\,. \tag{2.70}$$

Although one might superficially assume that the pole of the integrand is only half-encircled, the continuation of the integration contour into the lower complex plane actually makes it a full pole, and therefore the factor $1/(2\pi)$ of the ω integration is compensated by the factor 2π from the residue theorem.

Putting together Eqs. (2.21) and (2.69), we obtain the retarded Green function G_R,

$$G_R\,(\vec{r}-\vec{r}',t-t')=\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,f(\vec{k})\;P_{C_R}(t-t',k)\,, \tag{2.71}$$

where $k=|\vec{k}|$ and

$$f(\vec{k})=\frac{c}{2|\vec{k}|\epsilon_0}\;\exp\left(\mathrm{i}\,\vec{k}\cdot(\vec{r}-\vec{r}')\right)\,. \tag{2.72}$$

Then, we also use the result

$$P_{C_R}\,(t-t',k)=2\,\Theta\,(t-t')\;\sin(c\,k\,(t-t'))\,, \tag{2.73}$$

which we just derived. The Green function obtained with this path choice will be labelled G_R,

$$\boxed{\textbf{Retarded Green Function:}}\;\; G_R\,(\vec{r}-\vec{r}',t-t')=\Theta(t-t')\,\frac{c}{\epsilon_0}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{\mathrm{e}^{\mathrm{i}\,\vec{k}\cdot(\vec{r}-\vec{r}')}}{|\vec{k}|}\;\sin(ck\,(t-t'))\,. \tag{2.74}$$

Now we do the k integration in spherical $\vec{k}$ space, letting $\vec{r}-\vec{r}'$ be along the $\hat{\mathrm{e}}_z$ direction [$u_k=\cos(\theta_k)$]:

$$\begin{aligned}
G_R &= \Theta(t-t')\,\frac{c}{\epsilon_0\,(8\pi^3)}\int_0^\infty\mathrm{d}k\;k^2\int_0^{2\pi}\mathrm{d}\varphi_k\int_{-1}^{1}\mathrm{d}u_k\frac{\exp\left(\mathrm{i}\,k|\vec{r}-\vec{r}'|\;u_k\right)}{|\vec{k}|}\;\sin(c\,k\,(t-t')) \qquad (2.75)\\
&= \Theta(t-t')\,\frac{c\,(2\pi)}{\epsilon_0\,(8\pi^3)}\int_0^\infty\mathrm{d}k\;k^2\int_{-1}^{1}\mathrm{d}u_k\frac{\exp\left(\mathrm{i}\,k|\vec{r}-\vec{r}'|\;u_k\right)}{|\vec{k}|}\;\sin(c\,k\,(t-t'))\\
&= \Theta(t-t')\,\frac{c}{4\pi^2\epsilon_0}\int_0^\infty\mathrm{d}k\;k^2\,\frac{\exp\left(\mathrm{i}\,k\,|\vec{r}-\vec{r}'|\right)-\exp\left(-\,\mathrm{i}\,k|\vec{r}-\vec{r}'|\right)}{\mathrm{i}\,k^2|\vec{r}-\vec{r}'|}\;\sin(ck\,(t-t'))\\
&= \Theta(t-t')\,\frac{c}{4\pi^2\epsilon_0\,\mathrm{i}\,|\vec{r}-\vec{r}'|}\int_0^\infty\mathrm{d}k\;\left[\mathrm{e}^{\mathrm{i}\,k\,|\vec{r}-\vec{r}'|}\;\sin(ck\,(t-t'))+\mathrm{e}^{\mathrm{i}\,(-k)\,|\vec{r}-\vec{r}'|}\;\sin(c(-k)\,(t-t'))\right]
\end{aligned}$$

The integrand can be symmetrized,

$$\begin{aligned}
G_R &= \Theta(t-t')\,\frac{c}{4\pi^2\epsilon_0\,\mathrm{i}\,|\vec{r}-\vec{r}'|}\int_{-\infty}^{\infty}\mathrm{d}k\exp\left(\mathrm{i}\,k\,|\vec{r}-\vec{r}'|\right)\sin(ck(t-t'))\\
&= \Theta(t-t')\,\frac{c}{4\pi^2\epsilon_0\,\mathrm{i}\,|\vec{r}-\vec{r}'|}\int_{-\infty}^{\infty}\mathrm{d}k\,\left(\frac{1}{2\mathrm{i}}\right)\,\exp\left(\mathrm{i}\,k\,|\vec{r}-\vec{r}'|\right)\left[\mathrm{e}^{\mathrm{i}ck\,(t-t')}-\mathrm{e}^{-\mathrm{i}ck\,(t-t')}\right]\\
&= \Theta(t-t')\,\left(-\frac{c}{8\pi^2\epsilon_0|\vec{r}-\vec{r}'|}\right)\int_{-\infty}^{\infty}\mathrm{d}k\,\mathrm{e}^{\mathrm{i}\,k\,|\vec{r}-\vec{r}'|}\;\left[\mathrm{e}^{\mathrm{i}ck\,(t-t')}-\mathrm{e}^{-\mathrm{i}ck\,(t-t')}\right]\\
&= \Theta(t-t')\,\left(-\frac{c}{8\pi^2\,\epsilon_0\,|\vec{r}-\vec{r}'|}\right)\int_{-\infty}^{\infty}\mathrm{d}k\,\left[\mathrm{e}^{\mathrm{i}\,k\,(|\vec{r}-\vec{r}'|+c(t-t'))}-\mathrm{e}^{\mathrm{i}\,k\,(|\vec{r}-\vec{r}'|-c(t-t'))}\right]\,. \qquad (2.76)
\end{aligned}$$

The k integration is now trivial in view of the formula

$$\int_{-\infty}^{\infty}\mathrm{d}k\,\mathrm{e}^{\mathrm{i}kx}=2\pi\,\delta(x)\,. \tag{2.77}$$

The retarded Green function can thus be written as

$$
\begin{aligned}
G_R(\vec{r}-\vec{r}\,',t-t') &= -\frac{c}{4\pi\epsilon_0}\,\frac{\Theta(t-t')}{|\vec{r}-\vec{r}\,'|}\left\{\delta\Big(|\vec{r}-\vec{r}\,'|+c\,(t-t')\Big)-\delta\Big(|\vec{r}-\vec{r}\,'|-c\,(t-t')\Big)\right\} \\
&= +\frac{c}{4\pi\epsilon_0}\,\frac{\Theta(t-t')}{|\vec{r}-\vec{r}\,'|}\left\{\delta\Big(|\vec{r}-\vec{r}\,'|-c\,(t-t')\Big)-\delta\Big(|\vec{r}-\vec{r}\,'|+c\,(t-t')\Big)\right\} \\
&= \overline{G}_R(\vec{r}-\vec{r}\,',t-t')+\overline{\overline{G}}_R(\vec{r}-\vec{r}\,',t-t')\,, \qquad (2.78)
\end{aligned}
$$

where the two terms $\overline{G}_R(\vec{r}-\vec{r}\,',t-t')$ and $\overline{\overline{G}}_R(\vec{r}-\vec{r}\,',t-t')$ are identified as

$$
\overline{G}_R(\vec{r}-\vec{r}\,',t-t') = \frac{c}{4\pi\epsilon_0}\,\frac{\Theta(t-t')}{|\vec{r}-\vec{r}\,'|}\,\delta\Big(|\vec{r}-\vec{r}\,'|-c\,(t-t')\Big)\,, \qquad (2.79a)
$$

$$
\overline{\overline{G}}_R(\vec{r}-\vec{r}\,',t-t') = -\frac{c}{4\pi\epsilon_0}\,\frac{\Theta(t-t')}{|\vec{r}-\vec{r}\,'|}\,\delta\Big(|\vec{r}-\vec{r}\,'|+c\,(t-t')\Big) \qquad (2.79b)
$$

Another way to write this is

$$
\overline{G}_R(\vec{r}-\vec{r}\,',t-t') = \frac{c}{4\pi\epsilon_0}\,\frac{\Theta(t-t')}{|\vec{r}-\vec{r}\,'|}\,\delta\Big(|\vec{r}-\vec{r}\,'|-c\,(t-t')\Big)\,, \qquad (2.80a)
$$

$$
\overline{\overline{G}}_R(\vec{r}-\vec{r}\,',t-t') = \frac{(-c)}{4\pi\epsilon_0}\,\frac{1}{|\vec{r}-\vec{r}\,'|}\,\Theta(t-t')\,\delta\Big(|\vec{r}-\vec{r}\,'|-(-c)\,(t-t')\Big) = \overline{G}_R\Big|_{c\,\to\,-c}\,. \qquad (2.80b)
$$

The second term is derived from the first by a change in the sign of the speed of light. The term $\overline{\overline{G}}_R(\vec{r}-\vec{r}\,',t-t')$ is proportional to the product $\Theta(t-t')\,\delta\Big(|\vec{r}-\vec{r}\,'|+c\,(t-t')\Big)$. Unless $|\vec{r}-\vec{r}\,'|$ is zero or very close to zero, this term can be argued to vanish: Namely, the step function is nonvanishing only for $t>t'$, and conversely, for $c\,(t-t')>0$, even if $|\vec{r}-\vec{r}\,'|=0$, the argument of the Dirac-δ cannot vanish, which implies that the entire expression should vanish. However, as shown by an explicit calculation in [B. J. Wundt and U. D. Jentschura, *Sources, Potentials and Fields in Lorenz and Coulomb Gauge: Cancellation of Instantaneous Interactions for Moving Point Charges*, Ann. Phys. (N.Y.) **327**, pp. 1217–1230 (2012)], there is a caveat: Namely, if we think of the Dirac-δ function as a Gaussian with infinitesimal, but nonvanishing, width, then the nonvanishing part of the step function actually has a nonvanishing overlap with the representation of the Dirac-δ function. That infinitesimal overlap becomes important if we wish to *perform an integration by parts* in certain integrals, as considered in [B. J. Wundt and U. D. Jentschura, *Sources, Potentials and Fields in Lorenz and Coulomb Gauge: Cancellation of Instantaneous Interactions for Moving Point Charges*, Ann. Phys. (N.Y.) vol. 327, pp. 1217–1230 (2012)]. Indeed, if we need to integrate by parts, then the full representation $G_R=\overline{G}_R+\overline{\overline{G}}_R$ needs to be used. On the other hand, if no integrations by parts are needed, then the approximation

$$
\begin{aligned}
G_R(\vec{r}-\vec{r}\,',t-t') &\approx \overline{G}_R(\vec{r}-\vec{r}\,',t-t') \\
&= \frac{c}{4\pi\epsilon_0}\,\frac{\Theta(t-t')}{|\vec{r}-\vec{r}\,'|}\,\delta\Big(|\vec{r}-\vec{r}\,'|-c\,(t-t')\Big) \\
&\approx \frac{c}{2\pi\epsilon_0}\,\Theta(t-t')\,\delta\Big((\vec{r}-\vec{r}\,')^2-c^2\,(t-t')^2\Big) \qquad (2.81)
\end{aligned}
$$

is permissible. Here, the following formula

$$
\Theta(x)\;\delta[f(x)] = \sum_i \frac{1}{|f'(x_i)|}\,\Theta(x)\;\delta(x-x_i) = \sum_{x_i>0}\frac{1}{|f'(x_i)|}\,\Theta(x)\;\delta(x-x_i) \qquad (2.82)
$$

is used, which holds if all x_i with $f(x_i) = 0$ are positive. Indeed,

$$\Theta(\tau)\,\delta(\eta^2 - c^2\,\tau^2) = \Theta(\tau)\,\left[\frac{1}{2|\eta|}\,\delta(\eta - c\,\tau) + \frac{1}{2|\eta|}\,\delta(\eta + c\,\tau)\right] \approx \frac{\Theta(\tau)}{2\eta}\,\delta(\eta - c\,\tau)\,, \qquad \eta > 0\,. \tag{2.83}$$

Identifying $\eta = |\vec{r} - \vec{r}'|$ and $\tau = t - t'$, we can immediately understand the approximation (2.81). The retarded Green function thus constitutes a distribution centered exclusively on the light cone, i.e., it is vanishing except when the points $(\vec{r}, t)$ and $(\vec{r}', t')$ are separated by a distance that light travels in time $t - t'$,

$$\boxed{\textbf{Light Cone Condition:}} \qquad (\vec{r} - \vec{r}')^2 - c^2\,(t - t')^2 = 0\,. \tag{2.84}$$

With a grain of salt, we can now argue as follows: Since the step function $\Theta(t - t')$ is nonvanishing only for $t - t' > 0$, one of the Dirac δ functions always vanishes. We once more recall the simplified form of the retarded Green function,

$$\boxed{\textbf{Simple Retarded Green Function:}} \quad \overline{G}_R\,(\vec{r} - \vec{r}', t - t') = \frac{c}{4\pi\epsilon_0}\,\frac{\Theta(t - t')}{|\vec{r} - \vec{r}'|}\,\delta\Big(|\vec{r} - \vec{r}'| - c\,(t - t')\Big)\,. \tag{2.85}$$

The retarded Green function describes a disturbance (created at time t') propagating outward from a source point $\vec{r}'$. The condition $t - t' > 0$ is often interpreted as a causality condition. That is, the disturbance is detected at time t, which must be larger than the creation time, t', at the source. The Green function is also called a propagator as it "propagates" the disturbance from point $(\vec{r}', t')$ to the point $(\vec{r}, t)$ within the integral expression for the solution to the wave equation.

2.2.6 Advanced Green Function

The Green function obtained using the path C_A is similar, but is nonvanishing for $t < t'$. Note that one must repeat all the steps leading to the expression (2.78) for the retarded Green function but use the contour C_A, rather than C_R. The advanced Green function is obtained as

$$\begin{aligned} G_A\,(\vec{r} - \vec{r}', t - t') &= \frac{c}{4\pi\epsilon_0}\,\frac{1}{|\vec{r} - \vec{r}'|}\,\Theta(t' - t)\,\left\{\delta\Big(|\vec{r} - \vec{r}'| + c\,(t - t')\Big) - \delta\Big(|\vec{r} - \vec{r}'| - c\,(t - t')\Big)\right\} \\ &= \frac{c}{4\pi\epsilon_0}\,\frac{1}{|\vec{r} - \vec{r}'|}\,\Theta(t' - t)\,\left\{\delta\Big(|\vec{r} - \vec{r}'| - c\,(t' - t)\Big) - \delta\Big(|\vec{r} - \vec{r}'| + c\,(t' - t)\Big)\right\}. \end{aligned} \tag{2.86}$$

It is thus a distribution centered exclusively on the light cone. Writing $G_A\,(\vec{r} - \vec{r}', t - t') = \overline{G}_A\,(\vec{r} - \vec{r}', t - t') + \overline{\overline{G}}_A\,(\vec{r} - \vec{r}', t - t')$ with an obvious identification in view of Eq. (2.86), the following simplified expression is valid under the condition that manifestly $t < t'$, i.e., $G_A\,(\vec{r} - \vec{r}', t - t') \approx \overline{G}_A\,(\vec{r} - \vec{r}', t - t')$ with

$$\boxed{\textbf{Simple Advanced Green Function:}}$$

$$\begin{aligned} \overline{G}_A\,(\vec{r} - \vec{r}', t - t') &\approx \frac{c}{2\pi\epsilon_0}\,\frac{1}{|\vec{r} - \vec{r}'|}\,\Theta\,(t' - t)\;\;\delta\,(|\vec{r} - \vec{r}'| + c\,(t - t')) \\ &\approx \frac{c}{2\pi\epsilon_0}\,\Theta(t' - t)\,\delta\Big((\vec{r} - \vec{r}')^2 - c^2\,(t - t')^2\Big)\,. \end{aligned} \tag{2.87}$$

Here, $\overline{G}_A$ is defined in analogy to Eq. (2.80a). Note that one cannot use the simplified expressions when partial integrations are involved and surface terms have to be calculated; a deeper discussion on this point would go beyond our scope here. The advanced Green function corresponds to the retarded Green function under the replacement $t - t' \to t' - t$.

2.2.7 Feynman Contour Green Function

We recall the definition of the P_C function, which for the Feynman contour as given in Fig. 2.3 reads

$$P_{C_F}(t-t',|\vec{k}|) = \oint_{C_F} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\,\omega\,(t-t')} \left(\frac{1}{\omega + c\,|\vec{k}|} - \frac{1}{\omega - c\,|\vec{k}|} \right) . \tag{2.88}$$

The Feynman contour implies that the pole at $\omega = c\,|\vec{k}|$ is relevant for $t-t' > 0$, in which case it is encircled in the mathematically negative sense, whereas the pole at $\omega = -c\,|\vec{k}|$ is relevant for $t-t' < 0$, in which case it is encircled in the mathematically positive sense. Alternatively, we could thus write

$$\begin{aligned} P_{C_F}(t-t',|\vec{k}|) &= \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\,\omega\,(t-t')} \left(\frac{1}{\omega + c\,|\vec{k}| - \mathrm{i}\,\epsilon} - \frac{1}{\omega - c\,|\vec{k}| + \mathrm{i}\,\epsilon} \right) \\ &= \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\,\omega\,(t-t')} \left(\frac{1}{\omega + c\,|\vec{k}| - \mathrm{i}\,\epsilon|\vec{k}|} - \frac{1}{\omega - c\,|\vec{k}| + \mathrm{i}\,\epsilon|\vec{k}|} \right) , \end{aligned} \tag{2.89}$$

In the last step, we have introduced a redefinition of ϵ which ensures more favorable properties for the next steps in the calculation, which involves, in particular, a Fourier backtransformation to coordinate space. (The limit $\epsilon \to 0$ needs to be taken outside of the integral and is usually not indicated explicitly; the convergence is nonuniform and the limit cannot be pulled inside the integral.) So,

$$\begin{aligned} P_{C_F}\left(t-t',|\vec{k}|\right) &= -\frac{(-2\pi\mathrm{i})}{2\pi}\,\Theta\,(t-t') \left[\operatorname*{Res}_{\omega=ck-\mathrm{i}\epsilon k} \left(\frac{\mathrm{e}^{-\mathrm{i}\,\omega\left(t-t'\right)}}{\omega - c\,|\vec{k}|} \right) \right] \\ &\quad + \frac{2\pi\mathrm{i}}{2\pi}\,\Theta\,(-(t-t')) \left[\operatorname*{Res}_{\omega=-ck+\mathrm{i}\epsilon k} \left(\frac{\mathrm{e}^{-\mathrm{i}\,\omega\left(t-t'\right)}}{\omega + c\,|\vec{k}|} \right) \right] \\ &= \mathrm{i}\,\Theta\,(t-t')\;\mathrm{e}^{-\mathrm{i}(+ck-\mathrm{i}\epsilon k)(t-t')} + \mathrm{i}\,\Theta\,(-(t-t'))\;\mathrm{e}^{-\mathrm{i}(-ck+\mathrm{i}\epsilon k)(t-t')} \\ &= \mathrm{i}\,\Theta\,(t-t')\;\mathrm{e}^{-\mathrm{i}k(t-t')(c-\mathrm{i}\epsilon)} + \mathrm{i}\,\Theta\,(-(t-t'))\;\mathrm{e}^{\mathrm{i}k(t-t')(c-\mathrm{i}\epsilon)} . \end{aligned} \tag{2.90}$$

The second term equals the first under the replacement $t-t' \to t'-t$. The Green function obtained with this path choice will be labelled G_F,

Feynman Green Function:

$$G_F\,(\vec{r}-\vec{r}\,',t-t') = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{c\,\mathrm{e}^{\mathrm{i}\,\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{2\epsilon_0|\vec{k}|} \left(\mathrm{i}\,\Theta\,(t-t')\;\mathrm{e}^{-\mathrm{i}ck(t-t')} + \mathrm{i}\,\Theta\,(-(t-t'))\;\mathrm{e}^{\mathrm{i}ck(t-t')} \right) . \tag{2.91}$$

where we suppress for the moment the infinitesimal imaginary contribution to c. Now we do the k integration in spherical $\vec{k}$ space, letting $\vec{r}-\vec{r}\,'$ be along the $\hat{\mathrm{e}}_z$ direction $[u_k = \cos(\theta_k)]$. The Feynman Green function

G_F consists of two terms, the first of which reads

$$
\begin{aligned}
G_F^{(1)} &= \mathrm{i}\,\Theta(t-t')\,\frac{c}{2\,\epsilon_0\,(8\pi^3)}\int_0^\infty \mathrm{d}k\,k^2\int_0^{2\pi}\mathrm{d}\varphi_k\int_{-1}^{1}\mathrm{d}u_k\,\frac{\exp\left(\mathrm{i}\,k|\vec r-\vec r'|\,u_k\right)}{|\vec k|}\,\mathrm{e}^{-\mathrm{i}k(c-\mathrm{i}\epsilon)(t-t')} \\
&= \mathrm{i}\,\Theta(t-t')\,\frac{c\,(2\pi)}{2\,\epsilon_0\,(8\pi^3)}\int_0^\infty \mathrm{d}k\,k^2\int_{-1}^{1}\mathrm{d}u_k\,\frac{\exp\left(\mathrm{i}\,k|\vec r-\vec r'|\,u_k\right)}{|\vec k|}\,\mathrm{e}^{-\mathrm{i}k(c-\mathrm{i}\epsilon)(t-t')} \\
&= \mathrm{i}\,\Theta(t-t')\,\frac{c}{8\pi^2\epsilon_0}\int_0^\infty \mathrm{d}k\,k^2\,\frac{\exp\left(\mathrm{i}\,k\,|\vec r-\vec r'|\right)-\exp\left(-\mathrm{i}\,k|\vec r-\vec r'|\right)}{\mathrm{i}\,k^2|\vec r-\vec r'|}\,\mathrm{e}^{-\mathrm{i}k(c-\mathrm{i}\epsilon)(t-t')} \\
&= \mathrm{i}\,\Theta(t-t')\,\frac{c}{8\pi^2\epsilon_0\,\mathrm{i}\,|\vec r-\vec r'|}\int_0^\infty \mathrm{d}k\,\left[\mathrm{e}^{\mathrm{i}\,k\left(|\vec r-\vec r'|-c(t-t')+\mathrm{i}\,\epsilon'\right)}-\mathrm{e}^{-\mathrm{i}\,k\left(|\vec r-\vec r'|+c\,(t-t')-\mathrm{i}\,\epsilon'\right)}\right] \\
&= \mathrm{i}\,\Theta(t-t')\,\frac{c}{8\pi^2\epsilon_0\,\mathrm{i}\,|\vec r-\vec r'|}\left(\frac{1}{-\mathrm{i}\,\left(|\vec r-\vec r'|-c(t-t')+\mathrm{i}\,\epsilon'\right)}-\frac{1}{\mathrm{i}\,\left(|\vec r-\vec r'|+c(t-t')-\mathrm{i}\,\epsilon'\right)}\right) \\
&= \mathrm{i}\,\Theta(t-t')\,\frac{c}{8\pi^2\epsilon_0\,|\vec r-\vec r'|}\left(\frac{1}{|\vec r-\vec r'|-c(t-t')+\mathrm{i}\,\epsilon'}+\frac{1}{|\vec r-\vec r'|+c(t-t')-\mathrm{i}\,\epsilon'}\right) \\
&= \mathrm{i}\,\Theta(t-t')\,\frac{c}{4\pi^2\epsilon_0}\,\frac{1}{(\vec r-\vec r')^2-c^2\,(t-t')^2+\mathrm{i}\,\eta} \\
&= -\mathrm{i}\,\Theta(t-t')\,\frac{c}{4\pi^2\epsilon_0}\,\frac{1}{c^2\,(t-t')^2-(\vec r-\vec r')^2-\mathrm{i}\,\eta}\,,
\end{aligned}
\tag{2.92}
$$

where $\epsilon' = \epsilon\,(t-t')$ and $\eta = 2\,\epsilon'\,c\,(t-t') > 0$.

The second term for the Feynman Green function G_F reads

$$
\begin{aligned}
G_F^{(2)} &= \mathrm{i}\,\Theta\left(-(t-t')\right)\,\frac{c}{2\,\epsilon_0\,(8\pi^3)}\int_0^\infty \mathrm{d}k\,k^2\int_0^{2\pi}\mathrm{d}\varphi_k\int_{-1}^{1}\mathrm{d}u_k\,\frac{\exp\left(\mathrm{i}\,k|\vec r-\vec r'|\,u_k\right)}{|\vec k|}\,\mathrm{e}^{\mathrm{i}k(c-\mathrm{i}\epsilon)(t-t')} \\
&= \mathrm{i}\,\Theta\left(-(t-t')\right)\,\frac{c\,(2\pi)}{2\,\epsilon_0\,(8\pi^3)}\int_0^\infty \mathrm{d}k\,k^2\int_{-1}^{1}\mathrm{d}u_k\,\frac{\exp\left(\mathrm{i}\,k|\vec r-\vec r'|\,u_k\right)}{|\vec k|}\,\mathrm{e}^{\mathrm{i}k(c-\mathrm{i}\epsilon)(t-t')} \\
&= \mathrm{i}\,\Theta\left(-(t-t')\right)\,\frac{c}{8\pi^2\epsilon_0}\int_0^\infty \mathrm{d}k\,k^2\,\frac{\exp\left(\mathrm{i}\,k\,|\vec r-\vec r'|\right)-\exp\left(-\mathrm{i}\,k|\vec r-\vec r'|\right)}{\mathrm{i}\,k^2|\vec r-\vec r'|}\,\mathrm{e}^{\mathrm{i}k(c-\mathrm{i}\epsilon)(t-t')} \\
&= \mathrm{i}\,\Theta\left(-(t-t')\right)\,\frac{c}{8\pi^2\epsilon_0\,\mathrm{i}\,|\vec r-\vec r'|}\int_0^\infty \mathrm{d}k\,\left[\mathrm{e}^{\mathrm{i}\,k\left(|\vec r-\vec r'|+c(t-t')+\mathrm{i}\,\epsilon'\right)}-\mathrm{e}^{-\mathrm{i}\,k\left(|\vec r-\vec r'|-c\,(t-t')-\mathrm{i}\,\epsilon'\right)}\right] \\
&= \mathrm{i}\,\Theta\left(-(t-t')\right)\,\frac{c}{8\pi^2\epsilon_0\,\mathrm{i}\,|\vec r-\vec r'|}\left(\frac{1}{-\mathrm{i}\,\left(|\vec r-\vec r'|+c(t-t')+\mathrm{i}\,\epsilon'\right)}-\frac{1}{\mathrm{i}\,\left(|\vec r-\vec r'|-c(t-t')-\mathrm{i}\,\epsilon'\right)}\right) \\
&= \mathrm{i}\,\Theta\left(-(t-t')\right)\,\frac{c}{8\pi^2\epsilon_0\,|\vec r-\vec r'|}\left(\frac{1}{|\vec r-\vec r'|+c(t-t')+\mathrm{i}\,\epsilon'}+\frac{1}{|\vec r-\vec r'|-c(t-t')-\mathrm{i}\,\epsilon'}\right) \\
&= \mathrm{i}\,\Theta\left(-(t-t')\right)\,\frac{c}{4\pi^2\epsilon_0}\,\frac{1}{(\vec r-\vec r')^2-c^2\,(t-t')^2+\mathrm{i}\,\eta'} \\
&= -\mathrm{i}\,\Theta\left(-(t-t')\right)\,\frac{c}{4\pi^2\epsilon_0}\,\frac{1}{c^2\,(t-t')^2-(\vec r-\vec r')^2-\mathrm{i}\,\eta'}\,,
\end{aligned}
\tag{2.93}
$$

where $\epsilon' = -\epsilon(t-t') = \epsilon(t'-t) > 0$ and $\eta' = -2\,\epsilon'\,c\,(t-t') = 2\,\epsilon'\,c\,(t'-t) > 0$ in this case, because the prefactor contains $\Theta\left(-(t-t')\right)$. Adding the two contributions for $G_F = G_F^{(1)} + G_F^{(2)}$ and interpolating trivially for $t = t'$, we obtain the following,

Result for the Feynman Propagator:

$$
G_F(\vec r-\vec r', t-t') = -\mathrm{i}\,\frac{c}{4\pi^2\epsilon_0}\,\frac{1}{c^2\,(t-t')^2-(\vec r-\vec r')^2-\mathrm{i}\,\epsilon}\,.
\tag{2.94}
$$

Two important alternative forms of this propagator are as follows. First, because

$$
\frac{1}{g-\mathrm{i}\,\epsilon} = (\mathrm{P.V.})\,\frac{1}{g} + \mathrm{i}\,\pi\,\delta(g)\,,
\tag{2.95}
$$

we have

$$G_F(\vec{r}-\vec{r}',t-t') = -\mathrm{i}\,\frac{c}{4\pi^2\epsilon_0}\,(\mathrm{P.V.})\,\frac{1}{c^2\,(t-t')^2-(\vec{r}-\vec{r}')^2} + \frac{c}{4\pi\epsilon_0}\,\delta(c^2\,(t-t')^2-(\vec{r}-\vec{r}')^2)\,. \quad (2.96)$$

Another representation is

$$G_F(\vec{r}-\vec{r}',t-t') = -\mathrm{i}\,\frac{c}{8\pi^2\epsilon_0\,|\vec{r}-\vec{r}'|}\,\left(\frac{1}{c\,(t-t')-|\vec{r}-\vec{r}'|-\mathrm{i}\,\epsilon} - \frac{1}{c\,(t-t')+|\vec{r}-\vec{r}'|+\mathrm{i}\,\epsilon}\right)\,. \quad (2.97)$$

The poles are at

$$c\,(t-t') = |\vec{r}-\vec{r}'| + \mathrm{i}\,\epsilon\,, \qquad c\,(t-t') = -|\vec{r}-\vec{r}'| - \mathrm{i}\,\epsilon\,. \quad (2.98)$$

Now the Fourier transformation with respect to t of the Feynman Green function gives rise to the following integral, which we consider first of all only for real ω,

$$\begin{aligned} G_F(\vec{r}-\vec{r}',\omega) &= -\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}\tau\,\mathrm{e}^{\mathrm{i}\omega\tau}\,\frac{1}{8\pi^2\epsilon_0\,|\vec{r}-\vec{r}'|}\,\left(\frac{1}{\tau-|\vec{r}-\vec{r}'|/c-\mathrm{i}\,\epsilon} - \frac{1}{\tau+|\vec{r}-\vec{r}'|/c+\mathrm{i}\,\epsilon}\right) \\ &= \begin{cases} -\mathrm{i}\,2\pi\,\mathrm{i}\,\mathrm{e}^{\mathrm{i}\omega|\vec{r}-\vec{r}'|/c}\,\dfrac{1}{8\pi^2\epsilon_0\,|\vec{r}-\vec{r}'|} & \omega > 0\,, \\ -(-\mathrm{i})\,(-2\pi\,\mathrm{i})\,\mathrm{e}^{-\mathrm{i}\omega|\vec{r}-\vec{r}'|/c}\,\dfrac{1}{8\pi^2\epsilon_0\,|\vec{r}-\vec{r}'|} & \omega < 0\,, \end{cases} \\ &= \frac{\mathrm{e}^{\mathrm{i}\,\omega\,\mathrm{sgn}(\omega)|\vec{r}-\vec{r}'|/c}}{4\pi\epsilon_0\,|\vec{r}-\vec{r}'|}\,. \end{aligned} \quad (2.99)$$

Here, $\mathrm{sgn}(x)$ is the sign function, which equals $+1$ for $x > 0$ and -1 for $x < 0$. For complex ω, the Fourier integral would diverge either at $\tau = -\infty$ or at $\tau = +\infty$ and could not be directly calculated. The result (2.99) can be understood as follows: There are two poles in the τ integration, namely, at

$$\tau = \frac{|\vec{r}-\vec{r}'|}{c} + \mathrm{i}\epsilon\,, \qquad \tau = -\frac{|\vec{r}-\vec{r}'|}{c} - \mathrm{i}\epsilon\,. \quad (2.100)$$

If $\omega > 0$, then the τ integral needs to be closed in the upper half plane, i.e., the pole at $\tau = |\vec{r}-\vec{r}'|/c + \mathrm{i}\epsilon$ contributes. Conversely, if $\omega < 0$, then the τ integral needs to be closed in the upper half plane, i.e., the pole at $\tau = -|\vec{r}-\vec{r}'|/c - \mathrm{i}\epsilon$ contributes. For real ω,

$$G_F(\vec{r}-\vec{r}',\omega) = \frac{\mathrm{e}^{\mathrm{i}\,|\omega|\,|\vec{r}-\vec{r}'|/c}}{4\pi\epsilon_0\,|\vec{r}-\vec{r}'|}\,. \quad (2.101)$$

The question now is how to analytically continue the modulus function for manifestly complex ω. First of all, for real ω, we can write

$$\omega\,\mathrm{sgn}(\omega) = |\omega| = \sqrt{\omega^2}\,, \quad (2.102)$$

and this relation gives us a hint about how the analytic continuation needs to be done. The boundary condition in frequency space (convergence of the Fourier integral) translates into the condition that the contribution of frequencies of very large complex modulus to the Feynman propagator needs to vanish. This boundary condition can be implemented as follows. We first supply an infinitesimal imaginary part under the square root and write

$$\sqrt{\omega^2} \to \sqrt{\omega^2 + \mathrm{i}\,\epsilon}\,. \quad (2.103)$$

This is compatible with the requirement that $\sqrt{\omega^2} = |\omega|$ for real ω, no matter whether we lay the branch cut of the square root. We shall see in a moment that the most advantageous position for the branch cut is along the positive real axis, so that $\mathrm{Im}\,\sqrt{\omega^2+\mathrm{i}\epsilon} \geq 0$. Then,

$$G_F(\vec{r}-\vec{r}',\omega) = \frac{\exp\left(\mathrm{i}\,\sqrt{\omega^2+\mathrm{i}\epsilon}\,\dfrac{|\vec{r}-\vec{r}'|}{c}\right)}{4\pi\epsilon_0\,|\vec{r}-\vec{r}'|} \quad (2.104)$$

can be written as

$$G_F(\vec{r}-\vec{r}\,',\omega) = \frac{\exp\left(-b\,\dfrac{|\vec{r}-\vec{r}\,'|}{c}\right)}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|}\,, \qquad b = -\mathrm{i}\,\sqrt{\omega^2+\mathrm{i}\epsilon}\,, \qquad \mathrm{Re}(b) \overset{!}{\geq} 0\,, \qquad \mathrm{Im}\,\sqrt{\omega^2+\mathrm{i}\epsilon} \overset{!}{\geq} 0\,. \tag{2.105}$$

This condition is fulfilled if we choose the branch cut of the square root function along the positive real axis.

One can check, assuming ϵ (for the moment) to be a small, but finite quantity, that the argument of the square root function vanishes for $\sqrt{\omega^2+\mathrm{i}\epsilon}$, namely, the quantity $\omega^2+\mathrm{i}\epsilon$, vanishes for

$$\omega = \pm\sqrt{\frac{\epsilon}{2}}\,(1-\mathrm{i}) = \pm\frac{1-\mathrm{i}}{\sqrt{2}}\sqrt{\epsilon} = \pm\sqrt{\epsilon}\,\exp\left(-\frac{\mathrm{i}\pi}{4}\right). \tag{2.106}$$

If we define the branch cut of the square root function to be along the negative real axis, then the branch cuts, as a function of ω, extend to $\omega \to \pm\,\mathrm{i}\,\infty$, from the branch points. By contrast, if we define the branch cut of the square root function to be along the positive real axis, then branch cuts, as a function of ω, extend to $\omega \to \pm\,\infty$, from the branch points, and the latter possibility is the one that allows us to use the Feynman contour for the ω integration in the usual manner.

Along the Feynman contour (in this case, the real axis), one obtains, using (redefining ϵ)

$$\sqrt{\omega^2+\mathrm{i}\epsilon} \to \sqrt{\omega^2(1+\mathrm{i}\epsilon)} \to \sqrt{\omega^2} + \frac{1}{2}\,\mathrm{i}\,\sqrt{\omega^2}\epsilon = |\omega| + \mathrm{i}\,|\omega|\,\epsilon' \quad \to \quad |\omega|(1+\mathrm{i}\,\epsilon)\,. \tag{2.107}$$

We have the branch cut of the square root function along the positive real axis. The Fourier backtransformation, taking into account the cut of the square root function, reads as

$$\begin{aligned}
G_F(\vec{r}-\vec{r}\,',\tau) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \frac{\exp\left(\mathrm{i}\,\sqrt{\omega^2+\mathrm{i}\epsilon}\,\dfrac{|\vec{r}-\vec{r}\,'|}{c} - \mathrm{i}\omega\tau\right)}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|} \\
&= \int_0^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \frac{\exp\left(\mathrm{i}\,\omega\,\dfrac{|\vec{r}-\vec{r}\,'|}{c} - \mathrm{i}\omega\tau - \epsilon\,\omega\right)}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|} + \int_{-\infty}^{0} \frac{\mathrm{d}\omega}{2\pi}\, \frac{\exp\left(-\mathrm{i}\,\omega\,\dfrac{|\vec{r}-\vec{r}\,'|}{c} - \mathrm{i}\omega\tau + \epsilon\,\omega\right)}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|} \\
&= \int_0^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \frac{\exp\left[\left(\mathrm{i}\,\dfrac{|\vec{r}-\vec{r}\,'|}{c} - \mathrm{i}\,\tau - \epsilon\right)\,\omega\right]}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|} + \int_{-\infty}^{0} \frac{\mathrm{d}\omega}{2\pi}\, \frac{\exp\left[\left(-\mathrm{i}\,\dfrac{|\vec{r}-\vec{r}\,'|}{c} - \mathrm{i}\,\tau + \epsilon\right)\,\omega\right]}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|} \\
&= \int_0^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \frac{\exp\left[\left(\mathrm{i}\,\dfrac{|\vec{r}-\vec{r}\,'|}{c} - \mathrm{i}\,\tau - \epsilon\right)\,\omega\right]}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|} + \int_0^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \frac{\exp\left[\left(\mathrm{i}\,\dfrac{|\vec{r}-\vec{r}\,'|}{c} + \mathrm{i}\,\tau - \epsilon\right)\,\omega\right]}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|} \\
&= \frac{1}{8\pi^2\epsilon_0\,|\vec{r}-\vec{r}\,'|}\left(-\frac{1}{\mathrm{i}\,\dfrac{|\vec{r}-\vec{r}\,'|}{c} - \mathrm{i}\,\tau - \epsilon} - \frac{1}{\mathrm{i}\,\dfrac{|\vec{r}-\vec{r}\,'|}{c} + \mathrm{i}\,\tau - \epsilon}\right) \\
&= \frac{1}{8\pi^2\epsilon_0\,|\vec{r}-\vec{r}\,'|}\left(-\frac{\mathrm{i}\,c}{-|\vec{r}-\vec{r}\,'| + c\,\tau - \mathrm{i}\,\epsilon} - \frac{\mathrm{i}\,c}{-|\vec{r}-\vec{r}\,'| - c\,\tau - \mathrm{i}\,\epsilon}\right) \\
&= \frac{1}{8\pi^2\epsilon_0\,|\vec{r}-\vec{r}\,'|}\left(-\frac{\mathrm{i}\,c}{-|\vec{r}-\vec{r}\,'| + c\,\tau - \mathrm{i}\,\epsilon} + \frac{\mathrm{i}\,c}{|\vec{r}-\vec{r}\,'| + c\,\tau + \mathrm{i}\,\epsilon}\right) \\
&= -\,\frac{\mathrm{i}\,c}{8\pi^2\epsilon_0\,|\vec{r}-\vec{r}\,'|}\left(\frac{1}{c\,\tau - |\vec{r}-\vec{r}\,'| - \mathrm{i}\,\epsilon} - \frac{1}{c\,\tau + |\vec{r}-\vec{r}\,'| + \mathrm{i}\,\epsilon}\right) = -\mathrm{i}\,\frac{c}{4\pi^2\epsilon_0}\,\frac{1}{c^2\,\tau^2 - (\vec{r}-\vec{r}\,')^2 - \mathrm{i}\,\epsilon}\,.
\end{aligned} \tag{2.108}$$

When $\tau > 0$, the factor $\exp(-\mathrm{i}\,\omega\,\tau)$ vanishes for $\mathrm{Im}\omega \to -\infty$, and we can bend the left half of the contour to the lower half of the complex plane, directly below the cut. Alternatively, we can evaluate the Fourier backtransformation by evaluating the cut of the photon propagator,

$$
\begin{aligned}
G_F(\vec{r}-\vec{r}\,', \tau > 0) &= \int_{C_{\rm up}} \frac{\mathrm{d}\omega}{2\pi} \, \frac{\exp\left(\mathrm{i}\,\omega\,\dfrac{|\vec{r}-\vec{r}\,'|}{c} - \mathrm{i}\omega\tau - \epsilon\,\omega\right)}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|} - \int_{C_{\rm down}} \frac{\mathrm{d}\omega}{2\pi} \, \frac{\exp\left(-\mathrm{i}\,\omega\,\dfrac{|\vec{r}-\vec{r}\,'|}{c} - \mathrm{i}\omega\tau - \epsilon\,\omega\right)}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|} \\
&= \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \, \frac{\exp\left[\left(\mathrm{i}\,\dfrac{|\vec{r}-\vec{r}\,'|}{c} - \mathrm{i}\,\tau - \epsilon\right)\,\omega\right]}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|} - \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \, \frac{\exp\left[\left(-\mathrm{i}\,\dfrac{|\vec{r}-\vec{r}\,'|}{c} - \mathrm{i}\,\tau - \epsilon\right)\,\omega\right]}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|} \\
&= \frac{1}{8\pi^2\epsilon_0\,|\vec{r}-\vec{r}\,'|} \left(- \frac{1}{\mathrm{i}\,\dfrac{|\vec{r}-\vec{r}\,'|}{c} - \mathrm{i}\,\tau - \epsilon} + \frac{1}{-\mathrm{i}\,\dfrac{|\vec{r}-\vec{r}\,'|}{c} - \mathrm{i}\,\tau - \epsilon} \right) \\
&= \frac{1}{8\pi^2\epsilon_0\,|\vec{r}-\vec{r}\,'|} \left(- \frac{\mathrm{i}\,c}{-|\vec{r}-\vec{r}\,'| + c\,\tau - \mathrm{i}\,\epsilon} + \frac{\mathrm{i}\,c}{|\vec{r}-\vec{r}\,'| + c\,\tau - \mathrm{i}\,\epsilon} \right) \\
&= -\,\frac{\mathrm{i}\,c}{8\pi^2\epsilon_0\,|\vec{r}-\vec{r}\,'|} \left(\frac{1}{c\,\tau - |\vec{r}-\vec{r}\,'| - \mathrm{i}\,\epsilon} - \frac{1}{c\,\tau + |\vec{r}-\vec{r}\,'| - \mathrm{i}\,\epsilon} \right) \\
&= -\,\mathrm{i}\,\frac{c}{4\pi^2\epsilon_0} \, \frac{1}{c^2\,\tau^2 - (\vec{r}-\vec{r}\,')^2 - \mathrm{i}\,\epsilon\,\mathrm{sgn}(\tau)} = -\mathrm{i}\,\frac{c}{4\pi^2\epsilon_0} \, \frac{1}{c^2\,\tau^2 - (\vec{r}-\vec{r}\,')^2 - \mathrm{i}\,\epsilon}\,,
\end{aligned}
\tag{2.109}
$$

where the last line holds because we have assumed $\tau > 0$ in the first place.

Let us also carry out a full Fourier transformation with respect to space and time. We recall the definition of the Feynman propagator,

$$
G_F(\vec{r}-\vec{r}\,', \omega) = \frac{\exp\left(\mathrm{i}\,\sqrt{\omega^2 + \mathrm{i}\epsilon}\,\dfrac{|\vec{r}-\vec{r}\,'|}{c}\right)}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|}\,,
\tag{2.110}
$$

and calculate

$$
\begin{aligned}
G_F(\vec{k}, \omega) &= \int \mathrm{d}^3(\vec{r}-\vec{r}\,')\, G_F(\vec{r}-\vec{r}\,', \omega)\, \mathrm{e}^{-\mathrm{i}\,\vec{k}\cdot(\vec{r}-\vec{r}\,')} \\
&= \int \mathrm{d}^3\rho\; G_F(\vec{\rho}, \omega)\, \mathrm{e}^{-\mathrm{i}\,\vec{k}\cdot\vec{\rho}} = \int \mathrm{d}^3 r\; G_F(\vec{r}, \omega)\, \mathrm{e}^{-\mathrm{i}\,\vec{k}\cdot\vec{r}}\,.
\end{aligned}
\tag{2.111}
$$

Laying the axis of the $\vec{k}$ vector alongside the z axis, we can perform the calculation as

$$
\begin{aligned}
G_F(\vec{k}, \omega) &= \int \mathrm{d}^3 r\; G_F(\vec{r}, \omega)\, \mathrm{e}^{-\mathrm{i}\,\vec{k}\cdot\vec{r}} \\
&= 2\pi \int_0^\infty \mathrm{d}r\, r^2 \int_{-1}^{1} \mathrm{d}u\, \frac{\exp\left(\mathrm{i}\,\sqrt{\omega^2 + \mathrm{i}\epsilon}\,\dfrac{r}{c}\right)}{4\pi\epsilon_0\, r}\, \mathrm{e}^{-\mathrm{i}k\,r\,u} \\
&= \frac{1}{\epsilon_0} \int_0^\infty \mathrm{d}r\, \exp\left(\mathrm{i}\,\sqrt{\omega^2 + \mathrm{i}\epsilon}\,\frac{r}{c}\right)\, \frac{\sin(k\,r)}{k} \\
&= \frac{1}{\epsilon_0}\, \frac{1}{\vec{k}^2 - \omega^2/c^2 - \mathrm{i}\,\epsilon}\,.
\end{aligned}
\tag{2.112}
$$

The poles of the Fourier transform of the Feynman Green function are thus seen to be at $\omega = c|\vec{k}| - \mathrm{i}\epsilon$ and at $\omega = -c|\vec{k}| + \mathrm{i}\epsilon$, consistent with Fig. 2.2.

2.2.8 Summary of Green Functions

According to Eq. (2.87), the simple advanced Green function reads

$$\overline{G}_A\,(\vec{r}-\vec{r}\,',t-t') = \frac{c}{4\pi\epsilon_0}\,\frac{1}{|\vec{r}-\vec{r}\,'|}\,\Theta\,(t'-t)\;\delta\Big(|\vec{r}-\vec{r}\,'|+c\,(t-t')\Big)\,. \tag{2.113}$$

Its Fourier transform with respect to time is

$$G_A(\vec{r}-\vec{r}\,',\omega) = \frac{\mathrm{e}^{-\mathrm{i}\,\omega\,|\vec{r}-\vec{r}\,'|/c}}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|}\,, \tag{2.114}$$

and the full Fourier transform into frequency-wave-number space is

$$G_A(\vec{k},\omega) = \frac{c}{2|\vec{k}|\,\epsilon_0}\left(\frac{1}{\omega+c|\vec{k}|-\mathrm{i}\epsilon}-\frac{1}{\omega-c|\vec{k}|-\mathrm{i}\epsilon}\right) = \frac{1}{\epsilon_0}\,\frac{1}{\vec{k}^2-\omega^2/c^2+\mathrm{i}\epsilon\,\mathrm{sgn}[\mathrm{Re}(\omega)]}\,. \tag{2.115}$$

This result is immediately obvious from Eqs. (2.17), (2.20) and Fig. 2.1. If we desire to do Fourier integrals, then the following representation, with a rescaled $\epsilon \to |\omega|\epsilon$, is sometimes useful,

$$G_A(\vec{k},\omega) = \frac{1}{\epsilon_0}\,\frac{1}{\vec{k}^2-\omega^2/c^2+\mathrm{i}\epsilon\,\omega}\,, \tag{2.116}$$

in full analogy to Eq. (2.190) if we associate $\vec{k}^{\,2}$ with ω_0^2. In particular, $\epsilon > 0$, which is associated with the dampig term γ in Eq. (2.190), is a positive infinitesimal parameter). The simple form of the Retarded Green Function can be found in Eq. (2.85),

$$\overline{G}_R\,(\vec{r}-\vec{r}\,',t-t') = \frac{c}{4\pi\epsilon_0}\,\frac{1}{|\vec{r}-\vec{r}\,'|}\,\Theta(t-t')\;\delta\Big(|\vec{r}-\vec{r}\,'|-c\,(t-t')\Big)\,. \tag{2.117}$$

Its Fourier transform is

$$G_R(\vec{r}-\vec{r}\,',\omega) = \frac{\mathrm{e}^{\mathrm{i}\,\omega\,|\vec{r}-\vec{r}\,'|/c}}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|}\,. \tag{2.118}$$

The full Fourier transform into frequency-wave-number space is

$$G_R(\vec{k},\omega) = \frac{c}{2|\vec{k}|\,\epsilon_0}\left(\frac{1}{\omega+c|\vec{k}|+\mathrm{i}\epsilon}-\frac{1}{\omega-c|\vec{k}|+\mathrm{i}\epsilon}\right) = \frac{1}{\epsilon_0}\,\frac{1}{\vec{k}^2-\omega^2/c^2-\mathrm{i}\epsilon\,\mathrm{sgn}[\mathrm{Re}(\omega)]}\,. \tag{2.119}$$

If we desire to do Fourier integrals, then the following representation, with a rescaled $\epsilon \to |\omega|\epsilon$, is sometimes useful,

$$G_R(\vec{k},\omega) = \frac{1}{\epsilon_0}\,\frac{1}{\vec{k}^2-\omega^2/c^2-\mathrm{i}\epsilon\,\omega}\,, \tag{2.120}$$

in full analogy to Eq. (2.166) if we associate $\vec{k}^{\,2}$ with ω_0^2. The Feynman propagator (2.94) reads

$$G_F(\vec{r}-\vec{r}\,',t-t') = -\mathrm{i}\,\frac{c}{4\pi^2\epsilon_0}\,\frac{1}{c^2\,(t-t')^2-(\vec{r}-\vec{r}\,')^2-\mathrm{i}\,\epsilon}\,, \tag{2.121}$$

and its Fourier transform is

$$G_F(\vec{r}-\vec{r}\,',\omega) = \frac{\exp\left(\mathrm{i}\,\sqrt{\omega^2+\mathrm{i}\epsilon}\,\dfrac{|\vec{r}-\vec{r}\,'|}{c}\right)}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|}\,,\qquad \mathrm{Im}\sqrt{\omega^2+\mathrm{i}\epsilon}>0\,. \tag{2.122}$$

The latter condition clarifies how the branch cuts of the square root should be defined. The full Fourier transform is

$$G_F(\vec{k},\omega) = \frac{c}{2|\vec{k}|\,\epsilon_0}\left(\frac{1}{\omega+c|\vec{k}|-\mathrm{i}\epsilon}-\frac{1}{\omega-c|\vec{k}|+\mathrm{i}\epsilon}\right) = \frac{1}{\epsilon_0}\,\frac{1}{\vec{k}^2-\omega^2/c^2-\mathrm{i}\,\epsilon}\,. \tag{2.123}$$

2.3 Applications of the Retarded Green Function

2.3.1 Solution to the Wave Equation

We recall the wave equation (2.1) with sources in the general form

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right] \Psi\left(\vec{r}, t\right) = \frac{1}{\epsilon_0}\, F\left(\vec{r}, t\right) . \tag{2.124}$$

We also recall that the retarded Green function fulfills

$$\left(-\vec{\nabla}^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G_R\left(\vec{r} - \vec{r}', t - t'\right) = \frac{1}{\epsilon_0}\,\delta^3\left(\vec{r} - \vec{r}'\right)\,\delta(t - t') . \tag{2.125}$$

Using the retarded Green function solution, which we use in the simplified form $\overline{G}_R$ (no integration by parts), we may write a solution to the inhomogeneous wave equation as

$$\begin{aligned}
\Psi\left(\vec{r}, t\right) &= \Psi_{\rm hom}\left(\vec{r}, t\right) + \int \overline{G}_R\left(\vec{r} - \vec{r}', t - t'\right)\, F\left(\vec{r}', t'\right) \mathrm{d}^3 r'\, \mathrm{d}t' \\
&= \Psi_{\rm hom}\left(\vec{r}, t\right) + \frac{c}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \mathrm{d}^3 r' \int_{-\infty}^{\infty} \mathrm{d}t' \Theta\left(t - t'\right) \frac{1}{|\vec{r} - \vec{r}'|}\, \delta\Big(|\vec{r} - \vec{r}'| - c\left(t - t'\right)\Big)\, F\left(\vec{r}', t'\right) \\
&= \Psi_{\rm hom}\left(\vec{r}, t\right) + \frac{c}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \mathrm{d}^3 r' \int_{-\infty}^{\infty} \mathrm{d}t' \Theta\left(t - t'\right) \frac{1}{|\vec{r} - \vec{r}'|}\, \delta\left[c\left(\frac{|\vec{r} - \vec{r}'|}{c} - \left(t - t'\right)\right)\right] F\left(\vec{r}', t'\right) \\
&= \Psi_{\rm hom}\left(\vec{r}, t\right) + \frac{1}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \mathrm{d}^3 r' \int_{-\infty}^{\infty} \mathrm{d}t' \int \Theta\left(t - t'\right) \frac{1}{|\vec{r} - \vec{r}'|}\, \delta\left(t' - \left[t - \frac{|\vec{r} - \vec{r}'|}{c}\right]\right)\, F\left(\vec{r}', t'\right) \\
&= \Psi_{\rm hom}\left(\vec{r}, t\right) + \frac{1}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \mathrm{d}^3 r' \int_{-\infty}^{t} \mathrm{d}t' \frac{1}{|\vec{r} - \vec{r}'|}\, \delta\left(t' - \left[t - \frac{|\vec{r} - \vec{r}'|}{c}\right]\right)\, F\left(\vec{r}', t'\right) \mathrm{d}^3 r'\, \mathrm{d}t' \\
&= \Psi_{\rm hom}\left(\vec{r}, t\right) + \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3 r'\, \frac{1}{|\vec{r} - \vec{r}'|}\, F\left(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c}\right) .
\end{aligned} \tag{2.126}$$

Here, the expression

$$t_{\rm ret} = t - \frac{|\vec{r} - \vec{r}'|}{c} < t \tag{2.127}$$

is the retarded time, which expresses the fact that the electromagnetic perturbation propagates at the speed of light. The inequality $t_{\rm ret} < t$ ensures that the integration over t' from $-\infty$ to t always gives a nonvanishing result. In addition, $\Psi_{\rm hom}\left(\vec{r}, t\right)$ is a solution of the homogeneous wave equation.

Let us repeat the same steps for the advanced Green function,

$$
\begin{aligned}
\Psi\left(\vec{r},t\right) &= \Psi_{\rm hom}\left(\vec{r},t\right) + \int \overline{G}_A\left(\vec{r}-\vec{r}\,',t-t'\right)\; F\left(\vec{r}\,',t'\right) {\rm d}^3r'\,{\rm d}t' \\
&= \Psi_{\rm hom}\left(\vec{r},t\right) + \frac{c}{4\pi\epsilon_0}\int_{\mathbb{R}^3} {\rm d}^3r' \int_{-\infty}^{\infty} {\rm d}t'\Theta\left(t'-t\right) \frac{1}{|\vec{r}-\vec{r}\,'|}\,\delta\Big(|\vec{r}-\vec{r}\,'| + c\left(t-t'\right)\Big)\; F\left(\vec{r}\,',t'\right) \\
&= \Psi_{\rm hom}\left(\vec{r},t\right) + \frac{c}{4\pi\epsilon_0}\int_{\mathbb{R}^3} {\rm d}^3r' \int_{-\infty}^{\infty} {\rm d}t'\Theta\left(t'-t\right) \frac{1}{|\vec{r}-\vec{r}\,'|}\,\delta\left[c\left(\frac{|\vec{r}-\vec{r}\,'|}{c} + \left(t-t'\right)\right)\right]\; F\left(\vec{r}\,',t'\right) \\
&= \Psi_{\rm hom}\left(\vec{r},t\right) + \frac{1}{4\pi\epsilon_0}\int_{\mathbb{R}^3} {\rm d}^3r' \int_{-\infty}^{\infty} {\rm d}t' \int \Theta\left(t'-t\right) \frac{1}{|\vec{r}-\vec{r}\,'|}\,\delta\left(t' - \left[t + \frac{|\vec{r}-\vec{r}|}{c}\right]\right)\; F\left(\vec{r}\,',t'\right) \\
&= \Psi_{\rm hom}\left(\vec{r},t\right) + \frac{1}{4\pi\epsilon_0}\int_{\mathbb{R}^3} {\rm d}^3r' \int_{t}^{\infty} {\rm d}t' \frac{1}{|\vec{r}-\vec{r}\,'|}\,\delta\left(t' - \left[t + \frac{|\vec{r}-\vec{r}|}{c}\right]\right)\; F\left(\vec{r}\,',t'\right) {\rm d}^3r'\,{\rm d}t' \\
&= \Psi_{\rm hom}\left(\vec{r},t\right) + \frac{1}{4\pi\epsilon_0}\int_{\mathbb{R}^3} {\rm d}^3r' \frac{1}{|\vec{r}-\vec{r}\,'|}\, F\left(\vec{r}\,', t + \frac{|\vec{r}-\vec{r}\,'|}{c}\right)\,. \qquad (2.128)
\end{aligned}
$$

Here, the expression

$$t_{\rm adv} = t + \frac{|\vec{r}-\vec{r}\,'|}{c} > t \qquad (2.129)$$

is the advanced time, which expresses the fact that the electromagnetic perturbation propagates at the speed of light. The inequality $t_{\rm adv} > t$ ensures that the integration over t' from t to ∞ always gives a nonvanishing result. In addition, $\Psi_{\rm hom}\left(\vec{r},t\right)$ is a solution of the homogeneous wave equation.

2.3.2 Potentials and Sources in Coulomb Gauge

It is generally acknowledged that some mystery surrounds the so-called action-at-a-distance solution for the scalar potential which can be obtained in the Coulomb, or radiation, gauge. Due to the gauge condition in radiation gauge, the divergence of the vector potential vanishes, or, expressed differently, the vector potential is equal to its own transverse component [see Eq. (1.103)],

$$\vec{\nabla}\cdot\vec{A}_C\left(\vec{r},t\right) = 0\,, \qquad \vec{A}_C\left(\vec{r},t\right) = \vec{A}_{C\perp}\left(\vec{r},t\right)\,. \qquad (2.130)$$

In the following, we distinguish the scalar and vector potentials by the subscripts C for Coulomb gauge, and L for Lorenz gauge. In Coulomb gauge, the coupling to the sources is governed by the following equations [see Eq. (1.114)],

$$\vec{\nabla}^2\Phi_C\left(\vec{r},t\right) = -\frac{1}{\epsilon_0}\rho\left(\vec{r},t\right)\,, \qquad (2.131a)$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\vec{A}_{C\perp}\left(\vec{r},t\right) = \mu_0\,\vec{J}_\perp\left(\vec{r},t\right)\,, \qquad (2.131b)$$

$$-\epsilon_0\frac{\partial}{\partial t}\vec{E}_\parallel = \epsilon_0\frac{\partial}{\partial t}\vec{\nabla}\Phi_C\left(\vec{r},t\right) = \vec{J}_\parallel(\vec{r},t)\,. \qquad (2.131c)$$

We recall that the longitudinal and transverse components of a general vector field are analyzed in Eqs. (1.128) and (1.129). Equation (2.131a) couples the electrostatic potential to the source; it has the instantaneous, action-at-a-distance solution given in Eq. (1.121), which we recall for convenience,

$$\Phi_C(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int {\rm d}^3r'\,\frac{1}{|\vec{r}-\vec{r}\,'|}\,\rho(\vec{r}\,',t) + \Phi_{\rm hom}(\vec{r},t)\,, \qquad (2.132)$$

where $\Phi_{\rm hom}(\vec{r},t)$ is a solution to the homogeneous equation. The instantaneous coupling of the electrostatic potential to the charge density gives rise to a number of concerns. Our task is to show that the instantaneous character of the solution (2.132) does not lead to a contradiction with respect to the causality principle; an electromagnetic signal cannot travel faster than light and the fields (as opposed to the potentials) should be manifestly retarded.

The Lorenz condition for the scalar potential Φ and the vector potential $\vec{A}$ has been given in Eq. (1.84),

$$\vec{\nabla} \cdot \vec{A}_L\,(\vec{r},t) + \frac{1}{c^2}\,\frac{\partial}{\partial t}\Phi_L\,(\vec{r},t) = 0\,. \tag{2.133}$$

In Lorenz gauge, the scalar and vector potentials are coupled to the sources by inhomogeneous wave equations [see Eq. (1.92)],

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\,\Phi_L\,(\vec{r},t) = \frac{1}{\epsilon_0}\,\rho\,(\vec{r},t)\ , \tag{2.134a}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\,\vec{A}_L\,(\vec{r},t) = \mu_0\,\vec{J}\,(\vec{r},t)\ . \tag{2.134b}$$

We here provide two perspectives on the problem. The first perspective (see Sec. 2.3.3) relies on the fact that the solution of Eq. (2.131a) might otherwise indicate that the scalar potential in Coulomb gauge is non-retarded. However, once charges move, currents are generated in view of the continuity equation. The seemingly instantaneous scalar potential in Coulomb gauge is connected to the longitudinal part of the current density by Eq. (2.131c). It is instructive to observe that Eq. (2.131c) cannot alone provide to the solution of the solution of the problem; its divergence simply is the time derivative of Eq. (2.131a). However, Eq. (2.131c) shows that the situation is not so easy: We cannot argue that the seemingly instantaneous Coulomb interaction automatically gives rise to instantaneous fields; there is the additional condition (2.131c) which has to be fulfilled by the scalar potential. We shall see that the vector potential, in the Coulomb gauge, receives an additional contribution as compared to the Lorenz gauge. The additional term corresponds to the longitudinal part of the current density which has to be subtracted in order to obtain Eq. (2.131b). With the help of Eq. (2.131c), we are finally able to show that the supplementary term in the vector potential, in the Coulomb gauge, cancels the instantaneous contribution to the electric field and leads to manifestly retarded expressions.

Expressed differently, the additional constraint (2.131c) implies that the action-at-a-distance solution (2.132) is not universally a valid solution; it does not automatically fulfill Eq. (2.131c). Indeed, we shall see that the homogeneous term in Eq. (2.132) plays a crucial role in showing causality, together with Eq. (2.131c).

The second perspective (see Sec. 2.3.4) addresses the fact that the instantaneous instantaneous interaction integral can alternatively be written as a retarded integral, but with a different source term. Finally, in Sec. 2.3.5 (third perspective), we find an explicitly retarded expression for the Coulomb-gauge scalar potential, which fulfills both Eqs. (2.131a) as well as (2.131c).

2.3.3 Cancellation of the Instantaneous Term

According to Eq. (1.72), the electric field (in Coulomb gauge) is given as

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\Phi_C(\vec{r},t) - \frac{\partial}{\partial t}\vec{A}_{C\perp}(\vec{r},t)\,, \tag{2.135}$$

and therefore

$$\vec{E}_{\|}(\vec{r},t) = -\vec{\nabla}\Phi_C(\vec{r},t)\,, \qquad \vec{E}_{\perp}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{A}_{C\perp}(\vec{r},t)\,. \tag{2.136}$$

These components fulfill, explicitly, $\vec{\nabla} \times \vec{E}_{\parallel}(\vec{r}, t) = -\left(\vec{\nabla} \times \vec{\nabla}\right) \Phi_C(\vec{r}, t) = \vec{0}$, and $\vec{\nabla} \cdot \vec{E}_{\perp}(\vec{r}, t) = -\frac{\partial}{\partial t}\vec{\nabla} \cdot \vec{A}_{C\perp}(\vec{r}, t) = 0$. Therefore, the full longitudinal component of the electric field is given by $\vec{E}_{\parallel}(\vec{r}, t) = -\vec{\nabla}\Phi_C(\vec{r}, t)$, and the transverse component in Coulomb gauge is purely given by the time derivative of the transverse vector potential, without any "admixture" from the scalar potential. In Coulomb gauge, the longitudinal component of the electric field is calculated as

$$\vec{E}_{\parallel}(\vec{r}, t) = -\vec{\nabla}\Phi_C(\vec{r}, t) = -\frac{1}{4\pi\epsilon_0}\,\vec{\nabla}\int \mathrm{d}^3 r'\,\frac{1}{|\vec{r} - \vec{r}'|}\,\rho(\vec{r}', t) - \vec{\nabla}\Phi_{\mathrm{hom}}(\vec{r}, t)\,, \tag{2.137}$$

where Φ_{hom} is the solution to the homogeneous equation, adjusted so that Eq. (2.131c) is fulfilled.

The retarded Green function (2.78) enters the solutions of the Lorenz-gauge couplings given in Eqs. (2.134a) and (2.134b),

$$\Phi_L\left(\vec{r}, t\right) = \int \mathrm{d}^3 r'\,\mathrm{d}t'\,G_R(\vec{r} - \vec{r}', t - t')\,\rho\left(\vec{r}', t'\right)\,, \tag{2.138a}$$

$$\vec{A}_L\left(\vec{r}, t\right) = \frac{1}{c^2}\int \mathrm{d}^3 r'\,\mathrm{d}t'\,G_R(\vec{r} - \vec{r}', t - t')\,\vec{J}\left(\vec{r}', t'\right)\,. \tag{2.138b}$$

As has been stressed in Sec. 1.2.6, the potentials are not uniquely defined even within the family of potentials that fulfill the Coulomb gauge condition; a gauge re-transformation within the Coulomb gauge is possible according to Eq. (1.115). A permissible way to proceed is to use the retarded Green function to solve Eq. (2.131b), and to find the vector potential $\vec{A}_C(\vec{r}, t) = \vec{A}_{C\perp}(\vec{r}, t)$. The solution $\vec{A}_C(\vec{r}, t)$ can be written in terms of the Lorenz-gauge expression $\vec{A}_L(\vec{r}, t)$ and the supplementary term $\vec{A}_S(\vec{r}, t)$. We have,

$$\vec{A}_C(\vec{r}, t) = \frac{1}{c^2}\int \mathrm{d}^3 r'\int \mathrm{d}t'\;G_R(\vec{r} - \vec{r}', t - t')\;\vec{J}_{\perp}(\vec{r}', t') = \vec{A}_L(\vec{r}, t) + \vec{A}_S(\vec{r}, t)\,, \tag{2.139a}$$

$$\vec{A}_L(\vec{r}, t) = \frac{1}{c^2}\int \mathrm{d}^3 r'\int \mathrm{d}t'\;G_R(\vec{r} - \vec{r}', t - t')\;\vec{J}(\vec{r}', t')\,, \tag{2.139b}$$

$$\vec{A}_S(\vec{r}, t) = -\frac{1}{c^2}\int \mathrm{d}^3 r'\int \mathrm{d}t'\;G_R(\vec{r} - \vec{r}', t - t')\;\vec{J}_{\parallel}(\vec{r}', t')\,, \tag{2.139c}$$

where we have used the identity $\vec{J}_{\perp}(\vec{r}', t') = \vec{J}(\vec{r}', t') - \vec{J}_{\parallel}(\vec{r}', t')$. The supplementary term $\vec{A}_S(\vec{r}, t)$ is relevant to the Coulomb gauge and reads

$$\vec{A}_S(\vec{r}, t) = -\frac{1}{c^2}\int \mathrm{d}^3 r'\int \mathrm{d}t'\,G_R(\vec{r} - \vec{r}', t - t')\,\vec{J}_{\parallel}(\vec{r}', t')\,. \tag{2.140}$$

The time derivative of the supplementary term $\vec{A}_S(\vec{r}, t)$ contributes a supplementary term $\vec{E}_S$ to the electric field,

$$\begin{aligned}
\vec{E}_S(\vec{r}, t) = -\frac{\partial}{\partial t}\vec{A}_S(\vec{r}, t) &= \frac{1}{c^2}\int \mathrm{d}^3 r'\int \mathrm{d}t'\left[\frac{\partial}{\partial t}\,G_R(\vec{r} - \vec{r}', t - t')\right]\;\vec{J}_{\parallel}(\vec{r}', t') \\
&= -\frac{1}{c^2}\int \mathrm{d}^3 r'\int \mathrm{d}t'\left[\frac{\partial}{\partial t'}\,G_R(\vec{r} - \vec{r}', t - t')\right]\;\vec{J}_{\parallel}(\vec{r}', t') \\
&= \frac{1}{c^2}\int \mathrm{d}^3 r'\int \mathrm{d}t'\,G_R(\vec{r} - \vec{r}', t - t')\;\left[\frac{\partial}{\partial t'}\vec{J}_{\parallel}(\vec{r}', t')\right]\,,
\end{aligned} \tag{2.141}$$

where we have first transformed $\partial/\partial t \to -\partial/\partial t'$ and then used integration by parts to move the derivative on the current. Using Eq. (2.131c), this can be rewritten in terms of the potential as

$$
\begin{aligned}
E_S(\vec{r},t) &= \epsilon_0 \int \mathrm{d}^3 r' \int \mathrm{d}t' \, G_R(\vec{r}-\vec{r}\,',t-t') \, \vec{\nabla}' \left[\frac{1}{c^2} \frac{\partial^2}{\partial t'^2} \, \Phi_C\left(\vec{r}\,',t'\right) \right] \\
&= \epsilon_0 \, \vec{\nabla} \int \mathrm{d}^3 r' \int \mathrm{d}t' \, G_R(\vec{r},t,\vec{r}\,',t') \, \left[\left(\frac{1}{c^2} \frac{\partial^2}{\partial t'^2} - \vec{\nabla}'^{\,2} \right) + \vec{\nabla}'^{\,2} \right] \Phi_C\left(\vec{r}\,',t'\right) \\
&= \epsilon_0 \, \vec{\nabla} \int \mathrm{d}^3 r' \int \mathrm{d}t' \, \left[\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \right) G_R(\vec{r},t,\vec{r}\,',t') \right] \Phi_C\left(\vec{r}\,',t'\right) \\
&\quad + \epsilon_0 \, \vec{\nabla} \int \mathrm{d}^3 r' \int \mathrm{d}t' \, G_R(\vec{r},t,\vec{r}\,',t') \, \vec{\nabla}'^{2} \Phi_C\left(\vec{r}\,',t'\right) \, .
\end{aligned}
\tag{2.142}
$$

For the first term, we use partial integration twice, and we also take advantage of the symmetry properties of the retarded Green function. With Eqs. (2.131a) and (2.13), we find

$$
\begin{aligned}
E_S(\vec{r},t) &= \epsilon_0 \vec{\nabla} \int \mathrm{d}^3 r' \int \mathrm{d}t' \, \frac{1}{\epsilon_0} \delta^{(3)}\left(\vec{r}-\vec{r}\,'\right) \delta(t-t') \, \Phi_C\left(\vec{r}\,',t'\right) \\
&\quad - \epsilon_0 \vec{\nabla} \int \mathrm{d}^3 r' \int \mathrm{d}t' \, G_R(\vec{r},t,\vec{r}\,',t') \, \frac{1}{\epsilon_0} \rho\left(\vec{r}\,',t'\right) \\
&= \vec{\nabla} \Phi_C\left(\vec{r},t\right) - \vec{\nabla} \int \mathrm{d}^3 r' \int \mathrm{d}t' \, G_R(\vec{r},t,\vec{r}\,',t') \, \rho\left(\vec{r}\,',t'\right) \\
&= \vec{\nabla} \Phi_C\left(\vec{r},t\right) - \vec{\nabla} \Phi_L\left(\vec{r},t\right) \, .
\end{aligned}
\tag{2.143}
$$

In Coulomb gauge, the supplementary term $\vec{E}_S = -\partial_t \vec{A}_S$ due to the time derivative of the supplementary vector potential cancels the gradient of the Coulomb gauge scalar potential and adds the Lorenz gauge gradient of the scalar potential. Comparing the formulas for the electric field in Coulomb and Lorenz gauge, the following identity follows immediately,

$$
\begin{aligned}
\vec{E}_C(\vec{r},t) &= -\,\vec{\nabla}\Phi_C(\vec{r},t) - \frac{\partial}{\partial t}\vec{A}_C(\vec{r},t) = -\vec{\nabla}\Phi_C(\vec{r},t) - \frac{\partial}{\partial t}\left(\vec{A}_L(\vec{r},t) + \vec{A}_S(\vec{r},t)\right) \\
&= -\,\vec{\nabla}\Phi_C(\vec{r},t) - \frac{\partial}{\partial t}\vec{A}_L(\vec{r},t) + \left(\vec{\nabla}\Phi_C(\vec{r},t) - \vec{\nabla}\Phi_L(\vec{r},t)\right) \\
&= -\,\vec{\nabla}\Phi_L(\vec{r},t) - \frac{\partial}{\partial t}\vec{A}_L(\vec{r},t) = \vec{E}_L(\vec{r},t) \, .
\end{aligned}
\tag{2.144}
$$

We have temporarily denoted the "Coulomb gauge" electric field as $\vec{E}_C(\vec{r},t)$ and the "Lorenz gauge" electric field as $\vec{E}_L(\vec{r},t)$, even if both are actually equal due to gauge invariance, as shown.

Let us briefly summarize: In Coulomb gauge, there is an additional term $\vec{A}_S$ in the vector potential which is generated by the negative of the longitudinal component of the current density. The (negative of the) time derivative of the supplementary term in the vector potential yields an additional contribution to the electric field, in Coulomb gauge. The additional term in the electric field can be transformed into two parts, the first of which cancels the seemingly instantaneous electric field contribution in Coulomb gauge, obtained from the Coulomb-gauge electric potential, and the second yields the same result (the retarded one) as the gradient of the electric potential Φ_L in Lorenz gauge. In the end, the action-at-a-distance integral cancels, and the gauge invariance of the electric field is shown [Eq. (2.144)]. The overall conclusion is that in Coulomb gauge, in view of the condition (2.131c), the homogeneous solution Φ_{hom} in Eq. (2.132) has to be chosen so that the contribution of the action-at-a-distance term to the electric field cancels.

2.3.4 Longitudinal Electric Field as a Retarded Integral

We recall once more Eq. (2.137), which for the longitudinal component of the electric field reads as follows,

$$\vec{E}_\parallel(\vec{r},t) = -\vec{\nabla}\Phi_C(\vec{r},t) = -\frac{1}{4\pi\epsilon_0}\,\vec{\nabla}\int \mathrm{d}^3r'\,\frac{1}{|\vec{r}-\vec{r}'|}\,\rho(\vec{r}',t) - \vec{\nabla}\Phi_{\mathrm{hom}}(\vec{r},t)\,. \tag{2.145}$$

The first term in this expression has an action-at-a-distance form, which could in principle lead to a contradiction with respect to the causality principle, were it not for the additional constraint (2.131c), which implies the necessity of adding a suitable solution of the homogeneous equation. We recall that the decomposition of the electric field into longitudinal and transverse components is unique; an instantaneous character of the longitudinal component also would have disastrous consequences. We should thus investigate if, taking into account Eq. (2.131c), the longitudinal component of the electric field can alternatively be written as a manifestly retarded integral, for which we guess the form

$$\vec{E}_\parallel(\vec{r},t) = -\int \mathrm{d}^3r'\int \mathrm{d}t'\,G_R(\vec{r}-\vec{r}',t-t')\,\left(\frac{1}{c^2}\frac{\partial}{\partial t'}\vec{J}_\parallel(\vec{r}',t') + \vec{\nabla}'\rho(\vec{r}',t')\right)\,. \tag{2.146}$$

In this expression, we use Eq. (2.131c) in order to substitute for $\vec{J}_\parallel$ and Eq. (2.131a) in order to substitute for $\rho(\vec{r}',t')$,

$$\begin{aligned}\vec{E}_\parallel(\vec{r},t) =& -\int \mathrm{d}^3r'\int \mathrm{d}t' G_R(\vec{r}-\vec{r}',t-t')\bigg[\frac{1}{c^2}\frac{\partial}{\partial t'}\left(\epsilon_0\vec{\nabla}'\frac{\partial}{\partial t'}\Phi_C(\vec{r}',t')\right) \\ &+ \vec{\nabla}'\left(-\epsilon_0\,\vec{\nabla}'^2\Phi_C(\vec{r}',t')\right)\bigg] \\ =& -\int \mathrm{d}^3r'\int \mathrm{d}t'\,G_R(\vec{r},t,\vec{r}',t')\,\left(\frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \vec{\nabla}'^2\right)\,\epsilon_0\,\vec{\nabla}'\Phi_C(\vec{r}',t')\,.\end{aligned} \tag{2.147}$$

A double partial integration and use of Eq. (2.13) leads to the relation

$$\vec{E}_\parallel(\vec{r},t) = -\int \mathrm{d}^3r'\int \mathrm{d}t'\,\delta^{(3)}(\vec{r}-\vec{r}')\,\delta(t-t')\,\vec{\nabla}'\Phi_C(\vec{r}',t) = -\vec{\nabla}\Phi_C(\vec{r},t)\,, \tag{2.148}$$

which was to be shown.

In summary, we have demonstrated that the instantaneous integral for the longitudinal part of the electric field (2.137) can be rewritten as an integral involving the manifestly retarded Green function, with a nonstandard source term that does not only involve the charge density but also the longitudinal part of the current density. In the derivation, we have used Eq. (2.131c) which relates the charge density to the longitudinal part of the current density in Coulomb gauge.

2.3.5 Coulomb–Gauge Scalar Potential as a Retarded Integral

The last step in the analysis of the Coulomb gauge entails the calculation of the scalar potential, which is tantamount to finding an explicit expression for the homogeneous term Φ_C in Eq. (2.132). We start from Eq. (2.146), which we recall,

$$\vec{E}_\parallel(\vec{r},t) = -\int \mathrm{d}^3r'\int \mathrm{d}t'\,G_R(\vec{r}-\vec{r}',t-t')\,\left(\frac{1}{c^2}\frac{\partial}{\partial t'}\vec{J}_\parallel(\vec{r}',t') + \vec{\nabla}'\rho(\vec{r}',t')\right)\,. \tag{2.149}$$

Here, in view of Eq. (1.128), we can write the longitudinal component of the current density as the gradient of a scalar field $\mathcal{J}(\vec{r}',t')$,

$$\vec{J}_\parallel(\vec{r}',t') = \vec{\nabla}'\mathcal{J}(\vec{r}',t')\,. \tag{2.150}$$

So, the longitudinal component of the electric field becomes

$$\begin{aligned}
\vec{E}_{\|}(\vec{r},t) &= -\int \mathrm{d}^3 r' \int \mathrm{d}t'\, G_R(\vec{r}-\vec{r}',t-t')\, \vec{\nabla}' \left(\frac{1}{c^2}\frac{\partial}{\partial t'}\mathcal{J}(\vec{r}',t') + \rho(\vec{r}',t') \right) \\
&= \int \mathrm{d}^3 r' \int \mathrm{d}t'\, \vec{\nabla}' G_R(\vec{r}-\vec{r}',t-t') \left(\frac{1}{c^2}\frac{\partial}{\partial t'}\mathcal{J}(\vec{r}',t') + \rho(\vec{r}',t') \right) \\
&= -\vec{\nabla} \int \mathrm{d}^3 r' \int \mathrm{d}t'\, G_R(\vec{r}-\vec{r}',t-t') \left(\frac{1}{c^2}\frac{\partial}{\partial t'}\mathcal{J}(\vec{r}',t') + \rho(\vec{r}',t') \right) . \qquad (2.151)
\end{aligned}$$

Because $\vec{E}_{\|}(\vec{r},t) = -\vec{\nabla}\Phi_C(\vec{r},t)$, a valid ansatz for the scalar potential is

$$\Phi_C(\vec{r},t) = \int \mathrm{d}t' \int \mathrm{d}^3 r'\, G_R(\vec{r}-\vec{r}',t-t') \left(\frac{1}{c^2}\frac{\partial}{\partial t'}\mathcal{J}(\vec{r}',t') + \rho(\vec{r}',t') \right) , \qquad (2.152)$$

where $\mathcal{J}$ is given in Eq. (2.150).

The decisive step is to show that our ansatz (2.152) fulfills Eq. (2.131c),

$$\epsilon_0 \frac{\partial}{\partial t}\vec{\nabla}\Phi_C\,(\vec{r},t) = \vec{J}_{\|}(\vec{r},t)\,. \qquad (2.153)$$

The proof is relatively straightforward. One first integrates by parts,

$$\begin{aligned}
\epsilon_0 \frac{\partial}{\partial t}\vec{\nabla}\Phi_C\,(\vec{r},t) &= \epsilon_0 \frac{\partial}{\partial t}\vec{\nabla} \int \mathrm{d}t' \int \mathrm{d}^3 r' G_R(\vec{r}-\vec{r}',t-t') \left(\frac{1}{c^2}\frac{\partial}{\partial t'}\mathcal{J}(\vec{r}',t') + \rho(\vec{r}',t') \right) \\
&= \epsilon_0 \int \mathrm{d}t' \int \mathrm{d}^3 r'\, G_R(\vec{r}-\vec{r}',t-t') \left(\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\vec{\nabla}'\mathcal{J}(\vec{r}',t') + \frac{\partial}{\partial t'}\vec{\nabla}'\rho(\vec{r}',t') \right) . \qquad (2.154)
\end{aligned}$$

Use of Eq. (2.152) and of the continuity equation leads to

$$\begin{aligned}
\epsilon_0\, \partial_t \vec{\nabla}\Phi_C\,(\vec{r},t) &= \epsilon_0 \int \mathrm{d}t' \int \mathrm{d}^3 r' G_R(\vec{r}-\vec{r}',t-t') \left(\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\vec{J}_{\|}(\vec{r}',t') + \vec{\nabla}'\frac{\partial}{\partial t'}\rho(\vec{r}',t') \right) \\
&= \epsilon_0 \int \mathrm{d}t' \int \mathrm{d}^3 r'\, G_R(\vec{r}-\vec{r}',t-t') \left(\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\vec{J}_{\|}(\vec{r}',t') + \vec{\nabla}' \left(-\vec{\nabla}' \cdot \vec{J}_{\|}(\vec{r}',t') \right) \right) . \qquad (2.155)
\end{aligned}$$

This can be summarized as follows,

$$\epsilon_0\, \partial_t \vec{\nabla}\Phi_C\,(\vec{r},t) = \epsilon_0 \int \mathrm{d}t' \int \mathrm{d}^3 r'\, G_R(\vec{r}-\vec{r}',t-t') \left(\frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \vec{\nabla}'^2 \right) \vec{J}_{\|}(\vec{r}',t'). \qquad (2.156)$$

After a double partial integration, and use of Eq. (2.13), one can finally show that Eq. (2.152) fulfills Eq. (2.153). Because Eq. (2.152) is manifestly retarded, we have explicitly shown that the particular form of the scalar potential in Coulomb gauge, given in Eq. (2.152), does not lead to a contradiction with respect to the causality principle; the additional constraint (2.131c) ensures that the scalar potential is manifestly retarded.

2.4 Other Green Functions

2.4.1 Green Functions: The Paradigmatic Equations

It is very instructive here to verify the general concept on which a Green function is based, in terms of a few basic, illustrative equations. Let L be a linear differential operator. Let δ denote the Dirac-δ distribution.

Then, it is instructive to consider the

$$L\,G = \delta\,, \qquad L\,\psi = F\,, \tag{2.157}$$

where G is the Green functions, ψ is the "signal" and F is the "source". Once the Green function is found, the signal is determined by the equation

$$\psi = G \otimes F\,, \tag{2.158}$$

where $\otimes$ denotes the convolution. For functions of a single variable, the convolution is defined as follows,

$$(f \otimes g)(x) = \int \mathrm{d}x'\, f(x-x')\, g(x')\,. \tag{2.159}$$

The linear differential equation is solved in view of

$$L\,\psi = L\,G \otimes F = \delta \otimes F = F\,. \tag{2.160}$$

Let us examine the quantities in this formalism for the case of the Green function of the wave function. We have the following replacements/identifications

$$L \to \epsilon_0 \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \right)\,, \qquad \delta \to \delta^{(3)}(\vec{r}-\vec{r}')\,\delta(t-t')\,, \tag{2.161a}$$

$$(G \otimes F) \to \int \mathrm{d}^3 r' \int \mathrm{d}^3 t'\, G(\vec{r}-\vec{r}', t-t')\, F(\vec{r}', \vec{t}')\,. \tag{2.161b}$$

Green functions are powerful tools.

2.4.2 Green Function of the Harmonic Oscillator

We apply the Green function formalism to the harmonic oscillator. The "displacement" $x = x(t)$ of a damped, harmonic oscillator with unit mass $m = 1$ satisfies the equation

$$\ddot{x} + \gamma\,\dot{x} + \omega_0^2\, x = f\,(t)\ . \tag{2.162}$$

(a) Obtain the Green function $g = g(t-t')$ for this equation. Note that it has to fulfill the equation

$$\ddot{g}\,(t-t') + \gamma\,\dot{g}\,(t-t') + \omega_0^2\, g\,(t-t') = \delta\,(t-t')\ . \tag{2.163}$$

(c) Why is this Green function naturally obtained as the retarded Green function? Why do you have to change the sign of the damping term in order to obtain the advanced Green function?

(d) In the case that the "force" is given by $f\,(t) = f_0\,[\Theta\,(t) + \Theta\,(-t)\exp\,(t/\tau)]$, with vanishing boundary conditions in the infinite past, $x\,(-\infty) = 0$, and $\dot{x}\,(-\infty) = 0$, evaluate $x\,(t)$.

(e) Evaluate the work done by the driving force on the oscillator (per unit mass) in the time interval from $-\infty$ to $+\infty$ as a function of ω_0, γ, and τ.

(f) Now let $\gamma = \frac{1}{10}\omega_0$. Plot the work done by the driving force divided by the final (potential) energy stored in the oscillator as a function of $L = \ln\,(\omega_0\tau)$ from $L = -4$ to $L = 4$.

This problem is also a subject of an exercise. You may want to try and solve the above problem without considering the hints below. However, they might be useful as a check and guidance. It is up to you at which stage in the solution process you might wish to consult the hints.

Hints: We first want to understand why the Green function obtained is the retarded Green function. Let us consider Eq. (2.163) for a vanishing right-hand side $\delta(t-t')=0$ and $\omega_0=0$,

$$\ddot{g}(t-t')+\gamma\,\dot{g}(t-t')=0\,,\qquad g(t-t')=a+b\,\mathrm{e}^{-\gamma(t-t')}\,. \tag{2.164}$$

Here, a and b are integration constants. This solution is regular for $t-t'\to+\infty$ but grows exponentially for $t-t'\to-\infty$. The Green function $g(t-t')$ must necessarily be regular for both cases $t-t'\to\pm\infty$. The only way to implement the regularity condition is by way of the retarded Green function, which vanishes for $t-t'<0$. We write the Green function as a Fourier transform

$$g(t-t')=\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}\,\widetilde{g}(\omega)\,, \tag{2.165}$$

and show that

$$\widetilde{g}(\omega)=\frac{1}{\omega_0^2-\omega^2-\mathrm{i}\,\gamma\,\omega}=-\frac{1}{\omega^2+\mathrm{i}\,\gamma\,\omega-\omega_0^2}=-\frac{1}{(\omega-\omega_1)\,(\omega-\omega_2)}\,. \tag{2.166}$$

Note: Only regular functions (at $\pm\infty$) can be expanded into a Fourier series (Fourier integral); so our ansatz for the Green function already implies that the conditions of the above theorem are fulfilled. The poles are found at

$$\omega_1=\frac{1}{2}\left(-\sqrt{4\,\omega_0^2-\gamma^2}-\mathrm{i}\,\gamma\right)\,, \tag{2.167a}$$

$$\omega_2=\frac{1}{2}\left(\sqrt{4\,\omega_0^2-\gamma^2}-\mathrm{i}\,\gamma\right)\,. \tag{2.167b}$$

For $4\omega_0^2>\gamma^2$, both of these poles manifestly have a negative imaginary part. Show that

$$\begin{aligned}\operatorname*{Res}_{\omega=\omega_1}\left(-\frac{\mathrm{e}^{-\mathrm{i}\omega(t-t')}}{\omega^2+\mathrm{i}\,\gamma\,\omega-\omega_0^2}\right)&=\frac{\mathrm{e}^{-\gamma(t-t')/2}}{\sqrt{4\,\omega_0^2-\gamma^2}}\;\exp\left[-\frac{\mathrm{i}}{2}\sqrt{4\,\omega_0^2-\gamma^2}\,(t-t')\right]\,,\\ \operatorname*{Res}_{\omega=\omega_2}\left(-\frac{1}{\omega^2+\mathrm{i}\,\gamma\,\omega-\omega_0^2}\right)&=-\,\frac{\mathrm{e}^{-\gamma(t-t')/2}}{\sqrt{4\,\omega_0^2-\gamma^2}}\;\exp\left[\frac{\mathrm{i}}{2}\sqrt{4\,\omega_0^2-\gamma^2}\,(t-t')\right]\,.\end{aligned} \tag{2.168}$$

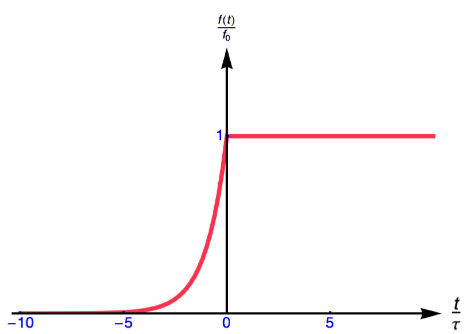


Figure 2.3: Illustration of the time-dependent force term for the harmonic oscillator, $f(t)=f_0\left[\Theta(t)+\Theta(-t)\exp(t/\tau)\right]$.

Close the contour in Eq. (2.165) in the lower half of the complex plane (why?) and show that

$$g(t-t')=g_R(t-t')=2\,\Theta(t-t')\,\mathrm{e}^{-\gamma(t-t')/2}\,\frac{\sin\left(\frac{1}{2}\sqrt{4\omega_0^2-\gamma^2}\,(t-t')\right)}{\sqrt{4\omega_0^2-\gamma^2}}\,. \tag{2.169}$$

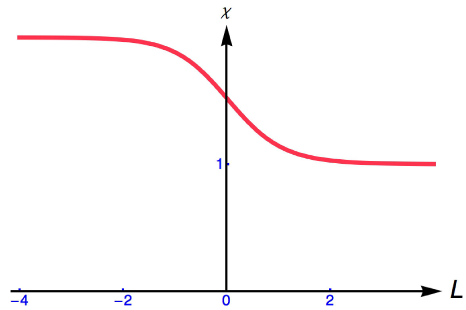


Figure 2.4: Plot of the ratio χ as defined in Eq. (2.187) for $\gamma = \omega_0/10$, with $X = \omega_0\,\tau = \exp(L)$.

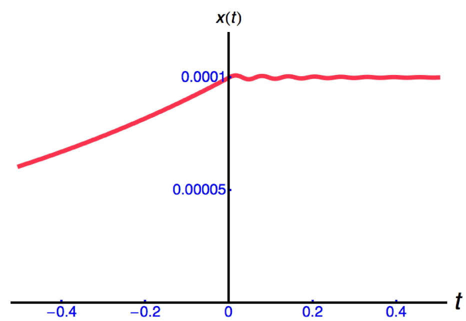


Figure 2.5: Plot of the trajectory $x(t)$ as obtained via the Green function formalism, for $\gamma = \omega_0/10$, $\omega_0 = 100$, and $\tau = 1$ as well as $f_0 = 1$.

This Green function fulfills

$$\ddot{g}\,(t-t') + \gamma\,\dot{g}\,(t-t') + \omega_0^2\,g\,(t-t') = \delta\,(t-t')\,. \tag{2.170}$$

In order to see this, one first carries out the differentiations with respect to time in Eq. (2.169) and verifies that the sum of the terms proportional to the step function vanish. That means that we have to differentiate the step function at least once in order to obtain a nonvanishing contribution. From the term $\dot{g}$, with $s = t - t'$ we have a term

$$T_0 = \left(\frac{\mathrm{d}}{\mathrm{d}s}\Theta(s)\right)\,\left[\left(2\,\mathrm{e}^{-\gamma s/2}\,\frac{\sin\left(\frac{1}{2}\,\sqrt{4\omega_0^2-\gamma^2}\,s\right)}{\sqrt{4\omega_0^2-\gamma^2}}\right)\right]\Bigg|_{s=0} = \delta(s)\times 0 = 0\,. \tag{2.171}$$

Then, one is left with the mixed term $(s = t - t')$ from the expression $\ddot{g}(t-t')$, which reads

$$T_1 = 2\times\left(\frac{\mathrm{d}}{\mathrm{d}s}\Theta(s)\right)\,\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(2\,\mathrm{e}^{-\gamma s/2}\,\frac{\sin\left(\frac{1}{2}\,\sqrt{4\omega_0^2-\gamma^2}\,s\right)}{\sqrt{4\omega_0^2-\gamma^2}}\right)\right]\Bigg|_{s=0} = 2\delta(s)\,. \tag{2.172}$$

The double derivative of the step function yields

$$T_2 = 2\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Theta(s)\right)\,\mathrm{e}^{-\gamma s/2}\,\frac{\sin\left(\frac{1}{2}\,\sqrt{4\omega_0^2-\gamma^2}\,s\right)}{\sqrt{4\omega_0^2-\gamma^2}} = 2\delta'(s)\,\mathrm{e}^{-\gamma s/2}\,\frac{\sin\left(\frac{1}{2}\,\sqrt{4\omega_0^2-\gamma^2}\,s\right)}{\sqrt{4\omega_0^2-\gamma^2}}\,. \tag{2.173}$$

In T_2, we may do an integration by parts. The reason is that the Dirac δ distribution only make sense as a distribution under an integral sign. Therefore, we may supply a test function $T(s)$ and replace

$$
\begin{aligned}
T_2 &= 2\delta'(s)\, \mathrm{e}^{-\gamma s/2} \, \frac{\sin\left(\frac{1}{2}\sqrt{4\omega_0^2-\gamma^2}\, s\right)}{\sqrt{4\omega_0^2-\gamma^2}} \, T(s) \qquad (2.174)\\
&\to -\delta(s) \left. \left[2\, \mathrm{e}^{-\gamma s/2} \, \frac{\sin\left(\frac{1}{2}\sqrt{4\omega_0^2-\gamma^2}\, s\right)}{\sqrt{4\omega_0^2-\gamma^2}} \, T'(s) \right] \right|_{s=0} \\
&\qquad - \delta(s) \left. \left[2 \, \frac{\mathrm{d}}{\mathrm{d}s} \left(\mathrm{e}^{-\gamma s/2} \, \frac{\sin\left(\frac{1}{2}\sqrt{4\omega_0^2-\gamma^2}\, s\right)}{\sqrt{4\omega_0^2-\gamma^2}} \right) \right] \right|_{s=0} T(s) \\
&= -\delta(s) \times 0 \times T'(s) - \delta(s) \times 1 \times T(s) = -\delta(s)\, T(s) \, .
\end{aligned}
$$

Evaluating the derivative in the latter term at $s=0$, we see that $T_2 = -\delta(s)$, and that

$$
\ddot{g}\,(t-t') + \gamma\, \dot{g}\,(t-t') + \omega_0^2 \, g\,(t-t') = T_0 + T_1 + T_2 = \delta\,(t-t') \, . \qquad (2.175)
$$

This verifies Eq. (2.170). We now investigate the force term

$$
\begin{aligned}
f\,(t') &= f_0 \, \Theta\,(t') + f_0 \, \Theta\,(-t') \, \exp\,(t'/\tau) = f_1(t') + f_2(t') \, , && (2.176a)\\
f_1(t') &= f_0 \, \Theta\,(t') \, , && (2.176b)\\
f_2(t') &= f_0 \, \Theta\,(-t') \, \exp\,(t'/\tau) \, . && (2.176c)
\end{aligned}
$$

and write

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} \mathrm{d}t' \, g_R(t-t') \, f(t') = x_1(t) + x_2(t) \, , && (2.177a)\\
x_1(t) &= \int_{-\infty}^{\infty} \mathrm{d}t' \, g_R(t-t') \, f_1(t') = f_0 \int_0^{\infty} \mathrm{d}t' \, g_R(t-t') \, , && (2.177b)\\
x_2(t) &= \int_{-\infty}^{\infty} \mathrm{d}t' \, g_R(t-t') \, f_2(t') = f_0 \int_{-\infty}^{0} \mathrm{d}t' \, g_R(t-t') \, \exp\,(t'/\tau) \, . && (2.177c)
\end{aligned}
$$

Term $x_1(t)$: Let us first consider the term $x_1(t)$. From Eq. (2.177b), we see that because $g_R(t-t')$ is proportional to $\Theta(t-t')$, the whole term $x_1(t)$ vanishes for $t<0$. Otherwise, for given t, the t' integration interval is reduced to $(0,t)$. An explicit calculation (your task!) shows that

$$
x_1(t) = \Theta(t) \, \frac{f_0}{\omega_0^2} \left[1 - \mathrm{e}^{-\gamma t/2} \left(\cos\left(\frac{1}{2}\sqrt{4\omega_0^2-\gamma^2}\, t\right) + \gamma \, \frac{\sin\left(\frac{1}{2}\sqrt{4\omega_0^2-\gamma^2}\, t\right)}{\sqrt{4\omega_0^2-\gamma^2}} \right) \right] \, , \qquad (2.178)
$$

which means that the damped oscillator approaches its final position f_0/ω_0^2 at $t\to\infty$ via an exponentially damped oscillatory motion. Please note the integrals

$$
\begin{aligned}
\int \exp(-a\,t) \, \sin(t) \, \mathrm{d}t &= -\frac{1}{a^2+1} \exp(-a\,t) \, [\cos(t) + a\,\sin(t)] \, , && (2.179a)\\
\int \exp(-a\,t) \, \cos(t) \, \mathrm{d}t &= -\frac{1}{a^2+1} \exp(-a\,t) \, [-a\,\cos(t) + \sin(t)] \, . && (2.179b)
\end{aligned}
$$

Term $x_2(t)$: For the term $x_2(t)$, we again have to differentiate two cases. If $t < 0$, then the integration interval in Eq. (2.177c) is restricted to $t' \in (-\infty, t)$. If $t > 0$, then the integration interval is $t' \in (-\infty, 0)$. Hence, an explicit calculation (your task!) shows that

$$
\begin{aligned}
x_2(t) = \Theta(t)\, & \frac{\mathrm{e}^{-\gamma t/2}\, f_0\, \tau\, \left[\tau\, \sqrt{4\omega_0^2-\gamma^2}\, \cos\left(\frac{1}{2}\sqrt{4\omega_0^2-\gamma^2}\, t\right) + (2+\gamma\,\tau)\, \sin\left(\frac{1}{2}\sqrt{4\omega_0^2-\gamma^2}\, t\right)\right]}{\sqrt{4\omega_0^2-\gamma^2}\, (1+\tau\,(\gamma+\tau\,\omega_0^2))} \\
& + \Theta(-t)\, \frac{\mathrm{e}^{t/\tau}\, f_0\, \tau^2}{1+\tau\,(\gamma+\tau\,\omega_0^2)} \,.
\end{aligned}
\tag{2.180}
$$

The full solution can then be presented as

$$
\begin{aligned}
x(t) = \Theta(t)\, f_0\, & \left(\frac{1}{\omega_0^2} - \frac{\mathrm{e}^{-\gamma t/2}\, (1+\gamma\tau)\, \cos\left(\frac{1}{2}\sqrt{4\omega_0^2-\gamma^2}\, t\right)}{\omega_0^2 (1+\gamma\tau+\tau^2\,\omega_0^2)} \right. \\
& \left. - \frac{\mathrm{e}^{-\gamma t/2}\, (\gamma+\gamma^2\,\tau - 2\omega_0^2\tau)\, \sin\left(\frac{1}{2}\sqrt{4\omega_0^2-\gamma^2}\, t\right)}{\omega_0^2\, \sqrt{4\omega_0^2-\gamma^2}\, (1+\gamma\tau\, + \omega_0^2\tau^2)} \right) + \Theta(-t)\, \frac{f_0\, \mathrm{e}^{t/\tau}\, \tau^2}{1+\tau\,(\gamma+\tau\,\omega_0^2)} \\
& = \Theta(t)\, x_+(t) + \Theta(-t)\, x_-(t) \,.
\end{aligned}
\tag{2.181}
$$

It fulfills

$$
x(t) \to 0 \quad \text{for} \quad t \to -\infty \,, \qquad x(t) \to \frac{f_0}{\omega_0^2} \quad \text{for} \quad t \to +\infty \,.
\tag{2.182}
$$

This is consistent with the boundary conditions naturally fulfilled by the retarded Green function, namely, vanishing boundary conditions in the infinite past. At $t = 0$, the two terms are in agreement at

$$
x(0) = \frac{f_0\, \tau^2}{1+\gamma\,\tau+\omega_0^2\tau^2} \,.
\tag{2.183}
$$

The work done on the oscillator is

$$
\begin{aligned}
W = \int \vec{f} \cdot \mathrm{d}\vec{s} &= \int_{-\infty}^{\infty} f(t)\, \dot{x}(t)\, \mathrm{d}t = \int_0^{\infty} f(t)\, \dot{x}_+(t)\, \mathrm{d}t + \int_{-\infty}^{0} f(t)\, \dot{x}_-(t)\, \mathrm{d}t \\
&= \frac{f_0^2 \tau^2}{2+2\gamma\,\tau+2\tau^2\,\omega_0^2} + \frac{f_0^2\, (1+\gamma\,\tau)}{\omega_0^2\, [1+\tau(\gamma+\omega_0^2\tau)]} = \frac{f_0^2}{\omega_0^2} - \frac{1}{2}\frac{f_0^2\tau^2}{1+\tau(\gamma+\omega_0^2\tau)} \,.
\end{aligned}
\tag{2.184}
$$

The energy stored in the harmonic oscillator is

$$
E = \frac{1}{2}\, \dot{x}^2 + \frac{1}{2}\, \omega_0^2 x^2 \,,
\tag{2.185}
$$

For $t \to +\infty$, the time derivative of the trajectory $x(t)$ vanishes, and $x(t) \to f_0/\omega_0^2$. So,

$$
E_\infty \to \frac{1}{2}\, \omega_0^2 \left(\frac{f_0}{\omega_0^2}\right)^2 = \frac{1}{2}\frac{f_0^2}{\omega_0^2} \qquad \text{for} \qquad t \to +\infty \,.
\tag{2.186}
$$

The ratio is

$$
\chi = \frac{W}{E_\infty} = 1 + \frac{1+\gamma\,\tau}{1+\tau(\gamma+\omega_0^2\tau)} \,.
\tag{2.187}
$$

For a relatively strongly damped system with $\gamma = \omega_0/10$, we have

$$
\chi = 1 + \frac{10+X}{10+X+10X^2} \,, \qquad X = \omega_0\, \tau \,, \qquad \gamma = \frac{\omega_0}{10} \,.
\tag{2.188}
$$

Setting $X = \exp(L)$, the dependence of χ on L is shown in Fig. 2.4. If the time scale τ during which the force is being switched on, is large on the time scale of the undamped oscillation, then $X = \omega_0\,\tau \gg 1$. Also, in that case, L is large, and the motion of the harmonic oscillator is only slightly damped. So, in that case, $\chi \to 1$. If the motion is strongly damped on the time scale of the undamped oscillation, then $\chi \to 2$, and half of the work invested in driving the oscillator is wasted into the damping friction force. An example of an only slightly damped motion with $\gamma = \omega_0/10$, $\omega_0 = 100$, $\tau = 1$, thus $X = \omega_0\,\tau = 100 \gg 1$, and $f_0 = 1$ is shown in Fig. 2.5.

A note on the advanced Green function. Under a time reversal operation $t \to -t$, the velocity of a particle in one-dimensional motion changes sign, but the accleration does not. The defining equation for the Green function under time reversal is obtained as follows,

$$\ddot{g_A}\,(t-t') - \gamma\,\dot{g_A}\,(t-t') + \omega_0^2\,g_A\,(t-t') = \delta\,(t-t')\,. \tag{2.189}$$

It differs from Eq. (2.163) in the sign of the damping term and can be obtained from Eq. (2.163) under the replacement $\gamma \to -\gamma$. The Fourier transform thus reads as

$$\widetilde{g}_A(\omega) = \frac{1}{\omega_0^2 - \omega^2 + \mathrm{i}\,\gamma\,\omega}\,. \tag{2.190}$$

Consequently, the advanced Green function of the damped harmonic oscillator reads as

$$\begin{aligned} g_A(t-t') &= \,-\,2\,\Theta(t'-t)\,\mathrm{e}^{\gamma(t-t')/2}\,\frac{\sin\left(\frac{1}{2}\,\sqrt{4\omega_0^2-\gamma^2}\,(t-t')\right)}{\sqrt{4\omega_0^2-\gamma^2}} \\ &= 2\,\Theta(t'-t)\,\mathrm{e}^{-\gamma(t'-t)/2}\,\frac{\sin\left(\frac{1}{2}\,\sqrt{4\omega_0^2-\gamma^2}\,(t'-t)\right)}{\sqrt{4\omega_0^2-\gamma^2}}\,. \end{aligned} \tag{2.191}$$

This propagator describes porpagation into the past, as does the advanced part of the Feynman propagator for quantized fields. One needs to reinterpret the time propagation direction in the advanced part of the propagator, in order to remain consistent with physical reality. A good way is to interpret the result as an amplitude in quantum mechanics, joining two space-time points the time coordinate of one of which happens to be before the time coordinate of the other.

A note on the concatenation approach. An alternative way of constructing the Green function is as follows. We first observe that the inhomogeneous and homogeneous equations, which we recall for convenience,

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}g\,(s) + \gamma\,\frac{\mathrm{d}}{\mathrm{d}s}g\,(s) + \omega_0^2\,g\,(s) = \delta\,(s)\,, \tag{2.192a}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}h\,(s) + \gamma\,\frac{\mathrm{d}}{\mathrm{d}s}h\,(s) + \omega_0^2\,h\,(s) = 0\,, \tag{2.192b}$$

are actually equivalent to each other except for the immediate vicinity of $s = 0$, because $\delta(s \neq 0) = 0$. Now, the general solution of the homogeneous equation can be easily determined and enters our ansatz for the Green function, which reads

$$\begin{aligned} g\,(s) = \Theta(s)\,&\left[A_+\,\mathrm{e}^{-\gamma\,s/2}\,\sin\left(\frac{1}{2}\,\sqrt{4\omega_0^2-\gamma^2}\,s\right)\right. \\ &\left.+B_+\,\mathrm{e}^{-\gamma\,s/2}\,\cos\left(\frac{1}{2}\,\sqrt{4\omega_0^2-\gamma^2}\,s\right)\right] \\ &+\Theta(-s)\,\left[A_-\,\mathrm{e}^{-\gamma\,s/2}\,\sin\left(\frac{1}{2}\,\sqrt{4\omega_0^2-\gamma^2}\,s\right)\right. \\ &\left.+B_-\,\mathrm{e}^{-\gamma\,s/2}\,\cos\left(\frac{1}{2}\,\sqrt{4\omega_0^2-\gamma^2}\,s\right)\right]\,. \end{aligned} \tag{2.193}$$

The integration constants $A_\pm$ and $B_\pm$ can be determined by *(i)* boundary conditions and *(ii)* integrating Eq. (2.192a) in an infinitesimal interval about $s = 0$. For the retarded Green function, we have to require that $A_- = B_- = 0$. This also follows from the regularity requirement at $s = -\infty$; the numerical value of the Green function would otherwise diverge in that limit. Furthermore, as the Green function needs to be continuous at $s = 0$, we have to impose the condition $B_+ = 0$; the cosine would otherwise induce a kink. After setting $A_- = B_- = B_+ = 0$, the remaining parameter A_+ can be determined by integrating Eq. (2.192a) in an infinitesimal interval around $s = 0$,

$$\begin{aligned} 1 = & \int_{-\epsilon}^{\epsilon} \delta(s)\, \mathrm{d}s = \int_{-\epsilon}^{\epsilon} \left[\frac{\mathrm{d}^2}{\mathrm{d}s^2} g(s) + \gamma\, \frac{\mathrm{d}}{\mathrm{d}s} g(s) + \omega_0^2\, g(s) \right] \mathrm{d}s \\ = & \left. \frac{\mathrm{d}}{\mathrm{d}s} g(s) \right|_{s=\epsilon} - \left. \frac{\mathrm{d}}{\mathrm{d}s} g(s) \right|_{s=-\epsilon} + \gamma\, [g(\epsilon) - g(-\epsilon)] + \omega_0^2\, g(0)\, \epsilon + \mathcal{O}(\epsilon^2) \\ = & \left. \frac{\mathrm{d}}{\mathrm{d}s} g(s) \right|_{s=\epsilon} - \left. \frac{\mathrm{d}}{\mathrm{d}s} g(s) \right|_{s=-\epsilon} + \mathcal{O}(\epsilon) = \left(\frac{1}{2}\, A_+ \sqrt{4\, \omega_0^2 - \gamma^2} \right) - (0) + \mathcal{O}(\epsilon)\, . \end{aligned} \tag{2.194}$$

We have assumed that the Green function itself is continuous while its derivative may have a kink. The result thus reads

$$A_+ = \frac{2}{\sqrt{4\, \omega_0^2 - \gamma^2}}\, . \tag{2.195}$$

Inserting this result into Eq. (2.193), we recover (2.169),

$$g(s) = 2\, \Theta(s)\, \mathrm{e}^{-\gamma s/2}\, \frac{\sin\left(\frac{1}{2} \sqrt{4\omega_0^2 - \gamma^2}\, s \right)}{\sqrt{4\omega_0^2 - \gamma^2}}\, . \tag{2.196}$$

We shall use the "concatenation approach" for the construction of the Green function in future derivations, related to electrodynamics.

2.4.3 Green Function and Atomic Polarizability

Let us try to connect our formalism as developed so far, to the properties of constituent atoms in a sample, combining the wisdom learned from the analysis of a Green function, with our picture of a damped harmonic oscillator, to see what we can learn about an atom in a sample, seen as a collection of driven damped harmonic oscillators, which describe the atomic transitions.

We will end up with a rather general model for the dielectric constant. In terms of the susceptibility $\chi_e(\omega)$, we have

$$\epsilon(\omega) = \epsilon_0\, \epsilon_r(\omega) = \epsilon_0\, \left(1 + \chi_e(\omega)\right)\, , \qquad \tilde{\vec{D}}(\vec{r}, \omega) = \epsilon(\omega)\, \tilde{\vec{E}}(\vec{r}, \omega) = \epsilon_r(\omega)\, \epsilon_0\, \tilde{\vec{E}}(\vec{r}, \omega)\, . \tag{2.197}$$

where $\epsilon_r(\omega)$ is a dimensionless relative polarizability. By definition, the dipole moment of an atom is related to its polarizability $\alpha(\omega)$ and to the applied electric field $\vec{E}(\omega)$ as follows (we assume spatially uniform fields)

$$\vec{p}_n = \sum_j q_{jn}\, \vec{r}_{jn} = \alpha(\omega)\, \vec{E}(\vec{r}, \omega)\, , \tag{2.198}$$

where we assume that $\alpha(\omega)$ is taylored to the nth atom. We had defined the polarization as

$$\vec{P}(\vec{r}, \omega) = \sum_n \vec{p}_n\, f(\vec{r} - \vec{r}_n) = \frac{N}{V} \vec{p}_n = \frac{N}{V}\, \alpha(\omega)\, \vec{E}(\vec{r}, \omega)\, , \tag{2.199}$$

where we assume that the test function $f(\vec{r}-\vec{r}_n)$ is a unit test function for the volume V, and there are N atoms in the test volume. Note that the normalization condition $\int d^3r\, f(\vec{r}-\vec{r}_n) = 1$ implies that f must have dimension of inverse volume. On the other hand,

$$\begin{aligned}\vec{D}(\vec{r},\omega) &= \epsilon_0 \left(\vec{E}(\vec{r},\omega) + \frac{1}{\epsilon_0}\vec{P}(\vec{r},\omega)\right) = \epsilon_0 \left(1 + \frac{1}{\epsilon_0}\frac{N}{V}\,\alpha(\omega)\right)\, \vec{E}(\vec{r},\omega) \\ &= \epsilon_0 \left(\vec{E}(\vec{r},\omega) + (\epsilon_r(\omega)-1)\,\vec{E}(\vec{r},\omega)\right)\,. \end{aligned} \tag{2.200}$$

Denoting by $\alpha(\omega)$ the dipole polarizability of the atom, we have

$$\epsilon_r(\omega) - 1 = \chi_e(\omega) = \frac{N_V}{\epsilon_0}\,\alpha(\omega)\,, \qquad N_V = \frac{N}{V}\,, \tag{2.201}$$

where N_V is the volume density of atoms. By convention, $\alpha(\omega) = \alpha_{\ell=1}(\omega)$ is the dipole ($2^{\ell=1}$-pole) dynamic polarizability of the ground state $|\phi_0\rangle$ of an atom. In terms of the oscillator strength $f_{n0} = f_{n0}^{(\ell=1)}$, we have a sum over virtual excited states $|\phi_n\rangle$,

$$\alpha(\omega) = \sum_n \frac{f_{n0}}{E_{n0}^2 - (\hbar\omega)^2}\,. \tag{2.202}$$

The energy differences between the ground state and the excited states are $E_{n0} = E_n - E_0$, where the sum over n also includes the continuous spectrum. The sum over n includes, in particular, all magnetic projections of the virtual states $|\phi_n\rangle$. The 2^ℓ-multipole oscillator strength is

$$f_{n0}^{(\ell)} = 2\,\frac{4\pi e^2}{(2\ell+1)^2}\,E_{n0}\sum_m\sum_{m_n}\left|\left\langle \phi_n\left|\sum_i (r_i)^\ell\,Y_{\ell m}(\hat{r}_i)\right|\phi_0\right\rangle\right|^2\,, \tag{2.203}$$

where we sum over the magnetic projections m of the spherical harmonic and over the magnetic projections m_n of the excited state $|\phi_n\rangle$. Furthermore, r_i is the radial coordinate of the ith electron, and $Y_{\ell m}(\hat{r}_i)$ is the spherical harmonic with the argument being equal to the unit vector of the position of the ith electron. The sum is over all electrons i whose position is $\vec{r}_i$ within the atomic system. The dipole oscillator strength is

$$f_{n0} = f_{n0}^{(\ell=1)}\,. \tag{2.204}$$

There is a sum rule,

$$\sum_n f_{n0} = Z\,e^2\,a_0^2\,E_h\,, \qquad a_0 = \frac{\hbar}{\alpha\,m_e\,c}\,, \qquad E_h = \alpha^2 m_e c^2\,. \tag{2.205}$$

Here, a_0 is the Bohr radius, E_h is the Hartree energy, and e is the elementary charge. Let us check units:

$$f_{n0} \sim (\mathrm{Cm})^2\,\mathrm{J}\,, \tag{2.206a}$$

$$\epsilon_0 \sim \frac{\mathrm{A\,s}}{\mathrm{V\,m}} = \frac{\mathrm{C}}{\mathrm{V\,m}}\,, \tag{2.206b}$$

$$\frac{N_V}{\epsilon_0}\,\frac{f_{n0}}{E_{n0}^2} \sim \frac{1}{\mathrm{m}^3}\,\frac{\mathrm{m\,V}}{\mathrm{C}}\,\frac{(\mathrm{Cm})^2}{\mathrm{J}^2}\,\mathrm{J} = \frac{\mathrm{J\,C}}{\mathrm{V}} = 1\,. \tag{2.206c}$$

We may restore the imaginary part of the virtual state energy according to

$$E_{n0} \to \mathrm{Re}\,E_{n0} + \mathrm{i}\,\mathrm{Im}E_{n0} = E_{n0} - \frac{\mathrm{i}}{2}\,\Gamma_n\,. \tag{2.207}$$

This corresponds to a spontaneous decay of the virtual state as

$$E_{n0} \to E_{n0} - \frac{\mathrm{i}}{2}\,\Gamma_n\,, \qquad \Gamma_n \ll E_{n0}\,, \tag{2.208}$$

where we have defined the Γ_n to be the imaginary part of the energy, as opposed to $\gamma_n = \Gamma_n/\hbar$, which has the physical dimension of frequency. So,

$$|\exp\left(-\mathrm{i}\,E_{n0}\,t\right)|^2 \to \left|\exp\left(-\mathrm{i}\,E_{n0}\,t - \frac{1}{2}\,\Gamma_n\,t\right)\right|^2 = \exp\left(-\Gamma_n\,t\right)\,. \tag{2.209}$$

In view of the identity

$$\frac{2E_{n0}}{E_{n0}^2 - (\hbar\omega)^2} = \frac{1}{E_{n0} - \hbar\omega} + \frac{1}{E_{n0} + \hbar\omega} \to \frac{1}{E_{n0} - \frac{\mathrm{i}}{2}\Gamma_n - \hbar\omega} + \frac{1}{E_{n0} - \frac{\mathrm{i}}{2}\Gamma_n + \hbar\omega}\,, \tag{2.210}$$

we can write $\alpha(\omega)$ as

$$\alpha(\omega) = \sum_n \frac{f_{n0}}{2E_{n0}}\left(\frac{1}{E_{n0} - \hbar\omega} + \frac{1}{E_{n0} + \hbar\omega}\right) \to \sum_n \frac{f_{n0}}{2E_{n0}}\left(\frac{1}{E_{n0} - \frac{\mathrm{i}}{2}\,\Gamma_n - \hbar\omega} + \frac{1}{E_{n0} - \frac{\mathrm{i}}{2}\,\Gamma_n + \hbar\omega}\right)\,. \tag{2.211}$$

We can approximate, neglecting terms of order Γ_n^2,

$$\frac{1}{2E_{n0}}\left(\frac{1}{E_{n0} - \mathrm{i}\frac{1}{2}\,\Gamma_n - \hbar\omega} + \frac{1}{E_{n0} - \mathrm{i}\frac{1}{2}\,\Gamma_n + \hbar\omega}\right) \approx \frac{1}{\left(E_{n0} - \mathrm{i}\frac{1}{2}\,\Gamma_n\right)^2 - (\hbar\omega)^2}\,. \tag{2.212}$$

Furthermore,

$$\frac{1}{\left(E_{n0} - \mathrm{i}\frac{1}{2}\,\Gamma_n\right)^2 - (\hbar\omega)^2} \approx \frac{1}{E_{n0}^2 - \mathrm{i}\,\Gamma_n\,E_{n0} - (\hbar\omega)^2} \approx \frac{1}{E_{n0}^2 - \mathrm{i}\,\Gamma_n\,\hbar\,\omega - (\hbar\omega)^2}\,. \tag{2.213}$$

The modification due to Γ_n is important only in the vicinity of the resonance. The corresponding expression for $\alpha(\omega)$ is

Atomic Polarizability as a Sum over Harmonic Oscillators:

$$\alpha(\omega) = \sum_n \frac{f_{n0}}{E_{n0}^2 - \mathrm{i}\,\Gamma_n\,\hbar\,\omega - (\hbar\omega)^2}\,. \tag{2.214}$$

Restoring prefactors, this can be written as

Relation of Dielectric Constant and Polarizability:

$$\tilde{\epsilon}(\omega) = \epsilon_0\left(1 + \frac{N_V}{\epsilon_0}\,\alpha(\omega)\right) = \epsilon_0\left(1 + \frac{N_V}{\epsilon_0}\,\sum_n \frac{f_{n0}}{E_{n0}^2 - \mathrm{i}\,\Gamma_n\,\hbar\,\omega - (\hbar\omega)^2}\right)\,. \tag{2.215}$$

If we scale variables according to $E_{n0} \to \hbar\omega_{n0}$ and $\Gamma_n \to \hbar\gamma_n$, then the analogy with a "collection of harmonic oscillators becomes obvious as comparison with Eq. (2.166) shows. Indeed, each term under the sum in Eq. (2.215) has a structure analogous to Eq. (2.166),

$$\widetilde{g}(\omega) = \frac{1}{\omega_0^2 - \mathrm{i}\,\gamma\,\omega - \omega^2}\,. \tag{2.216}$$

Let us briefly discuss an application of the outlined formalism to the famous plasma oscillations, at least within the free-electron gas approximation. First, we observe in Eq. (2.215) that, restricting the sum over n to only a single intermediate state, say $n = 1$, we can replace and define, as appropriate,

$$\frac{N_V}{\epsilon_0}\,\frac{f_{n0}}{\hbar^2} \to \omega_p^2\,, \qquad \frac{E_{n0}}{\hbar} \to \omega_0\,, \qquad \frac{\Gamma_n}{\hbar} \to \gamma\,. \tag{2.217}$$

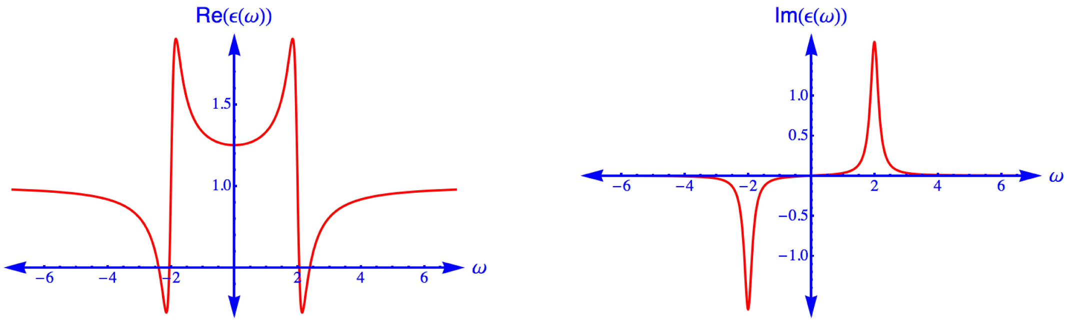


Figure 2.6: Plot of $\mathrm{Re}(\epsilon_r(\omega))$ and $\mathrm{Im}(\epsilon_r(\omega))$ for $\omega_p = 1$, $\omega_0 = 2$, and $\gamma = 0.3$, according to Eq. (2.218). The functional shape of $\mathrm{Re}(\epsilon_r(\omega))$ is consistent with in-phase driving below resonance, with a phase jump by π as one crosses the resonance. The deviation of $\mathrm{Re}(\epsilon_r(\omega))$ from unity at high ω is proportional to $1/\omega^2$.

Then,

$$\tilde{\epsilon}(\omega) = \epsilon_0 \left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - \mathrm{i}\,\omega\,\gamma} \right) . \tag{2.218}$$

This is a simple model for a dielectric constant, inspired by a simple harmonic oscillator model (see also Fig. 2.6). The model has the right dimension, as the harmonic oscillator Green function relates the displacement (the position, or "dipole moment") of the driven oscillator to the perturbation, or force, or, electric field.

Let us try to dig a little deeper into this analogy. A free electron gas consists of nearly free electrons, i.e., with a small restorative force, so that the positions of the electrons are described by the formula

$$\ddot{x} + \gamma\,\dot{x} + \omega_0^2\,x = \frac{e}{m_e}\,E_{\rm ext}(t) \tag{2.219}$$

where x is the (collective) displacement of the (free) electrons in the gas, e is the (physical) electron charge, m_e is the electron mass, and $E_{\rm ext}(t)$ is the (externally applied) electric field.

In Fourier space, the displacement $\widetilde{x}(\omega)$ is thus given by

$$\widetilde{x}(\omega) = \frac{1}{\omega_0^2 - \omega^2 - \mathrm{i}\,\omega\,\gamma}\,\frac{e}{m_e}\,\widetilde{E}_{\rm ext}(\omega) . \tag{2.220}$$

If N_V is the volume density of plasma electrons, then the volume density of the polarization is given by the formula

$$\widetilde{P}(\omega) = N_V\,e\,\widetilde{x}(\omega) = \frac{1}{\omega_0^2 - \omega^2 - \mathrm{i}\,\omega\,\gamma}\left(\frac{N_V\,e^2}{m_e}\right)\widetilde{E}_{\rm ext}(\omega) . \tag{2.221}$$

Now,

$$\widetilde{\epsilon}(\omega) = \frac{\epsilon_0\,\widetilde{E}_{\rm ext}(\omega) + \widetilde{P}(\omega)}{\widetilde{E}_{\rm ext}(\omega)} = \epsilon_0\left[1 + \frac{1}{\omega_0^2 - \omega^2 - \mathrm{i}\,\omega\,\gamma}\left(\frac{N_V\,e^2}{\epsilon_0\,m_e}\right)\right] = \epsilon_0\left[1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - \mathrm{i}\,\omega\,\gamma}\right] , \tag{2.222}$$

where we have identified the plasma frequency ω_p as follows,

$$\omega_p^2 = \frac{N_V\,e^2}{\epsilon_0\,m_e} . \tag{2.223}$$

We note the correspondence of Eqs. (2.222) and (2.218). Furthermore, because the electron gas is free, the internal resonance frequency $\omega_0 \ll \omega_p$ is very nearly equal to zero on the scale of the plasma frequency, and we have the

$$\boxed{\text{Drude Model of the Dielectric Constant:}} \qquad \widetilde{\epsilon}(\omega) \approx \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2 + \mathrm{i}\,\omega\,\gamma} \right) . \tag{2.224}$$

This formula is canonically known as the Drude model for the dielectric constant of a free electron gas. The so-called plasma model would do away with the term $\mathrm{i}\,\omega\,\gamma$, which would result in an expression which is in conflict with the dispersion relations that have to be fulfilled by the dielectric constant. Basically, the paradigm of these dispersion relations (the so-called Kramers–Kronig relations) is that you cannot have an nontrivial real part in a dielectric constant without also having an imaginary part; dispersion and absorption go hand in hand.

One can also derive the formula for ω_p by different means. Let us consider a bulk material of electrons, with volume density N_V, moving "freely" relative to a jellium of ionized cores, by a collective distance x_c. Let A be the (large) cross-sectional area of the box, and $V = A\,x_c$ its volume. The induced charges of the electrons, "sticking out" at either end of the box, are easily calculated as follows,

$$Q_1 = N_V\, e\, A\, x_c , \qquad Q_2 = -Q_1 \,. \tag{2.225}$$

By Gauss's theorem in integrated form, applied to the layer in between the charged structure, we have for the induced electric field $\oint \vec{E}_{\mathrm{ind}} \cdot \mathrm{d}\vec{A} = (\epsilon_0)^{-1} \int \mathrm{d}^3 r\, \rho(\vec{r})$, we have

$$E_{\mathrm{ind}}\, A = -\frac{1}{\epsilon_0}\, N_V\, e\, A\, x_c \,, \qquad E_{\mathrm{ind}} = -\frac{1}{\epsilon_0}\, N_V\, e\, x_c \,, \tag{2.226}$$

where the sign follows from a geometric consideration (exercise!). We now express the dielectric constant as follows,

$$\widetilde{\epsilon}(\omega) \approx \epsilon_0\, \frac{\widetilde{E}_{\mathrm{ext}}(\omega) - \widetilde{E}_{\mathrm{ind}}(\omega)}{\widetilde{E}_{\mathrm{ext}}(\omega)} \,. \tag{2.227}$$

Since the electrons in the bulk medium are free, we can approximate

$$m_e\, \frac{\mathrm{d}^2 x_c}{\mathrm{d}t^2} \approx e\, E_{\mathrm{ext}}(t) \,, \qquad -\omega^2\, \frac{m_e}{e}\, \widetilde{x}_c(\omega) \approx \widetilde{E}_{\mathrm{ext}}(\omega) \,, \tag{2.228}$$

where in the latter step we go into Fourier space. Finally,

$$\widetilde{\epsilon}(\omega) \approx \epsilon_0\, \frac{-\omega^2\, \dfrac{m_e}{e}\, \widetilde{x}_c(\omega) + \dfrac{1}{\epsilon_0}\, N_V\, e\, \widetilde{x}_c(\omega)}{-\omega^2\, \dfrac{m_e}{e}\, \widetilde{x}_c(\omega)} = \epsilon_0 \left(1 - \frac{N_V\, e^2}{\epsilon_0\, m_e}\, \frac{1}{\omega^2} \right) = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2} \right) . \tag{2.229}$$

This "plasma" model ignores the possibility of damping, as given in Eq. (2.224), and, as we later see, does not fulfill the Kramers–Kronig relations, in view of the fact that $\widetilde{\epsilon}(\omega)$ cannot be real everywhere along the real ω-axis. Our model (2.224) thus is physically more consistent than the "plasma" or "bulk" formula (2.229).

2.4.4 Scalar Wave and Phase Shifts

The formulation in terms of the phenomenological Maxwell equations enables us to express the propagation of electromagnetic waves in a medium in terms of the permittivity and permeability functions, $\epsilon(\omega)$ and $\mu(\omega)$. These lead to a phase slip of a wave propagation in the medium in comparison to the same wave when traveling in vacuum. Here, in general terms, we relate the phase slip to the change in the propagation velocity, i.e., its wavelength (the frequency stays the same) and the attenuation coefficient.

We shall consider, in general, the relation between phase shift and propagation velocity of a wave described by a scalar particle, through a medium. This is not a wave described by a vector potential, like a standard electromagnetic wave, but could be a matter wave describing a quantal particle, perhaps. We thus have a harmonic plane, scalar wave propagating through a medium of identical scatterers/absorbers, say in the $+z$ direction. We will investigate the change in the phase of the wave between the plane z and the plane $z+\Delta z$.

The (angular) frequency of the wave is ω_0 and the vacuum wave number is k_0. In the plane given by a constant z coordinate, the wave is given by

$$\Psi\,(z,t) = \psi_0\ \exp\left[\mathrm{i}\ (k_0\ z - \omega_0\ t + \phi\,(z))\right]\,, \tag{2.230}$$

where $\phi\,(z)$ is the "phase shift" or the "phase slip" on the wave function because the wave is travelling in a medium. Our goal here is to relate the infinitesimal changes in the phase slip about z to the change in the propagation velocity, and the damping constant in the medium, in a general setting. If there were no medium between z and $z+\Delta z$, the wave transmitted through the slab of thickness Δz would be

$$\Psi_0\,(z+\Delta z,t) = \psi_0\ \exp\left[\mathrm{i}\ (k_0\,(z+\Delta z) - \omega_0\ t + \phi\,(z))\right]\,. \tag{2.231}$$

However, it is really given by

$$\Psi\,(z+\Delta z,t) = \psi_0\ \exp\left[\mathrm{i}\ (k_0\,(z+\Delta z) - \omega_0\ t + \phi\,(z+\Delta z))\right]\,, \tag{2.232}$$

because we also have to take into account the change in the additional phase $\phi(z)$ in going from z to $z+\Delta z$. Assuming a small Δz, we now expand,

$$\begin{aligned}
\Psi\,(z+\Delta z,t+\Delta t) &= \psi_0\ \exp\left[\mathrm{i}\ (k_0\ \ (z+\Delta z) - \omega_0\ (t+\Delta t) + \phi\,(z+\Delta z))\right] \\
&\approx \psi_0\ \exp\left[\mathrm{i}\ (k_0\,(z+\Delta z) - \omega_0\ (t+\Delta t) + \phi\,(z)) + \mathrm{i}\,\frac{\mathrm{d}\phi\,(z)}{\mathrm{d}z}\Delta z\right] \\
&= \psi_0\ \exp\left[\mathrm{i}\ (k_0\,(z+\Delta z) - \omega_0\ (t+\Delta t) + \phi\,(z))\right]\ \exp\left[\mathrm{i}\,\frac{\mathrm{d}\phi\,(z)}{\mathrm{d}z}\ \Delta z\right] \\
&= \Psi(z,t)\ \exp\Big[\mathrm{i}\ (k_0\ \Delta z - \omega_0\ \Delta t)\Big]\ \exp\left[\mathrm{i}\,\frac{\mathrm{d}\phi\,(z)}{\mathrm{d}z}\ \Delta z\right] \\
&\equiv \Psi(z,t)\ \exp\Big[\mathrm{i}\ (k_0\ \Delta z - \omega_0\ \Delta t)\Big]\ \exp\left[\gamma\ \mathrm{e}^{\mathrm{i}\theta}\ \Delta z\right] \\
&= \Psi(z,t)\ \exp\Big[\mathrm{i}\ (k_0\ \Delta z - \omega_0\ \Delta t)\Big]\ \exp\Big[\gamma\cos\theta\ \Delta z + \mathrm{i}\,\gamma\ \sin\theta\ \Delta z\Big] \\
&= \Psi(z,t)\ \exp\Big[\mathrm{i}\ ((k_0+\gamma\ \sin\theta)\ \ \Delta z - \omega_0\ \Delta t)\Big]\ \exp\Big(\gamma\cos\theta\ \Delta z\Big)\,.
\end{aligned} \tag{2.233}$$

We have defined implicitly, at the place marked with the "$\equiv$" sign,

$$\mathrm{i}\,\frac{\mathrm{d}\phi\,(z)}{\mathrm{d}z}\ \Delta z \equiv \gamma\ \mathrm{e}^{\mathrm{i}\theta}\ \Delta z\,. \tag{2.234}$$

That means that the wave has the same phase at points Δz reached in time Δt as at point z at time t if the following equation is fulfilled,

$$(k_0+\gamma\ \sin\theta)\ \ \Delta z - \omega_0\ \Delta t = 0\,, \qquad\qquad v_{\mathrm{med}} = \frac{\Delta z}{\Delta t} = \frac{\omega_0}{k_0+\gamma\ \sin\theta}\,. \tag{2.235}$$

The latter equation defines the phase velocity in the medium.

The expression $\gamma \ \exp(\mathrm{i}\,\theta)$ depends on the density of scattering centers and the interactions between the wave and the scatterers. On a microscopic scale, the "scattered wave" is generated as follows. The incident wave perturbs the electronic structure of the atoms and molecules, and this rearrangement produces an emitted electromagnetic wave which adds to the transmitted wave. This assumes that there is a sufficient density of scatterers to generate a plane wave. We assume that the scattered wave is due to a response of the scatterer to the incident wave. In this case we expect that the scattered wave will lag the incident wave in time so θ is expected to be positive. In addition, in order to provide destructive interference, $\pi/2 < \theta \leq \pi$, in order to describe exponential damping rather than exponential divergence of the wave amplitude.

The exponential damping factor then is (in the case of attenuation, for an optical gain medium the sign is reversed)

$$\exp(z\ \gamma\ \cos(\theta)) = \exp(-\gamma\ |\cos\ \theta|\ z)\ . \tag{2.236}$$

The wave is damped, with a damping constant $\gamma\,|\cos(\theta)|$, as it propagates through the medium. The speed of the wave in the medium is

$$v_{\text{med}} = \frac{\omega_0}{k_0 + \gamma \sin(\theta)} = \frac{v_{\text{vac}}}{1 + (\gamma/k_0)\ \sin(\theta)} < v_{\text{vac}}\ . \tag{2.237}$$

Note that the range $\pi/2 < \theta \leq \pi$ now provides an exponential damping $[\cos(\theta) < 0]$ and a reduced phase velocity, as given by the relations $\sin(\theta) > 0$ and $v_{\text{med}} < v_{\text{vac}}$. This example for a scalar wave indicates how several of the features of the electromagnetic wave in a medium can be explained.

Measurements.—The above discussion indicates how the wavelength dependence of the propagation and scattering of the electromagnetic wave can provide information concerning the structure of a system. In particular, x-ray diffraction is commonly used to determine the structure parameters of liquids and solids. These measurements are particularly sensitive to the electron density function. In our discussion, we have assumed that the frequency of the waves will not drive the system at a resonance. This would have complicated the discussion of the macroscopic equations.. However, the frequency dependence of the propagation, scattering, and absorption of the electromagnetic waves provides information concerning the dynamics of a system (the frequency response function for the system). These measurements of the wavelength and frequency dependence of the interaction of the electromagnetic waves and a system are generally done with harmonic, continuous waves (cw).

Another class of measurements, which have become more common with the advent of lasers, are those in which an electromagnetic wave excites a system to an initial state and the evolution of the system is probed using a second wave. The availability of pulsed lasers, with pulses less than a picosecond (even of the order of 10 femtoseconds), provides the opportunity to follow the evolution of a system on a time scale on the order of a picosecond.

Chapter 3

Electromagnetic Radiation from Oscillatory Sources

3.1 Orientation

This chapter is devoted to the radiation emitted from oscillating current distributions at a specific frequency ω. The mixed representation of the Green function of the wave equation (position-frequency representation) will be used. The multipole decomposition will be treated.

3.2 Basic Formulas

3.2.1 Helmholtz Equation and Green Function

A simple radiating system consists of a localized charge density $\rho(\vec{r}, t)$ and a localized current density $\vec{J}(\vec{r}, t)$. The system is considered to be localized if its dimensions are small compared to the wavelength of the radiation. We will consider radiating sources in a vacuum and begin with the equations for the vector and scalar potentials in the Lorenz gauge and SI units,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\Phi(\vec{r}, t) = \frac{1}{\epsilon_0}\,\rho(\vec{r}, t)\ , \tag{3.1a}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\vec{A}(\vec{r}, t) = \mu_0\,\vec{J}(\vec{r}, t)\ . \tag{3.1b}$$

The first of these equations can be written

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\frac{1}{c}\Phi(\vec{r}, t) = \frac{1}{\epsilon_0 c^2}\,c\,\rho(\vec{r}, t) = \mu_0\,c\,\rho(\vec{r}, t)\ . \tag{3.2}$$

The 4-vectors are (four-vector potential and four-current density)

$$A^\mu = \left(\Phi, c\,\vec{A}\right) \qquad \text{and} \qquad J^\mu = \left(\rho, \frac{1}{c}\,\vec{J}\right)\ . \tag{3.3}$$

The consistency of the physical units of the entities in the first 4-vector can be verified immediately, as follows: The electric field has the dimension of the expression $\vec{\nabla}\Phi$. The magnetic field has the dimension of $\vec{\nabla} \times \vec{B}$. As for travelling waves, and certainly in terms of physical units, $|\vec{E}| \sim c\,|\vec{B}|$, the factor c is explained. Charge density has units of

$$[\rho] = \frac{[Q]}{[r]^3} = \frac{[Q]}{[t]\;[r]^2}\;\frac{[t]}{[r]} = \frac{[\vec{J}]}{[c]}\,, \tag{3.4}$$

which explains the units in the second 4-vector given in Eq. (3.3).

Alternatively, we can write Eq. (3.1) in the following form, which contains the vacuum permittivity ϵ_0 and is amenable to the solution via the retarded Green function,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\Phi\left(\vec{r},t\right) = \frac{1}{\epsilon_0}\;\rho\left(\vec{r},t\right)\,, \tag{3.5a}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)c\,\vec{A}\left(\vec{r},t\right) = \frac{\mu_0\,c^2}{c}\;\vec{J}\left(\vec{r},t\right) = \frac{1}{\epsilon_0}\;\left(\frac{1}{c}\;\vec{J}\left(\vec{r},t\right)\right)\,. \tag{3.5b}$$

In order to analyze a radiating system, we use the retarded Green function for the basic wave equation, in the simplified form given in Eq. (2.85) which is valid when, manifestly, $t > t'$,

$$\begin{aligned} G_R\left(\vec{r}-\vec{r}',t-t'\right) &= \Theta\left(t-t'\right)\;\frac{c}{4\pi\epsilon_0|\vec{r}-\vec{r}'|}\;\delta\left(|\vec{r}-\vec{r}'| - c\left(t-t'\right)\right) \\ &= \Theta\left(t-t'\right)\;\frac{1}{4\pi\epsilon_0|\vec{r}-\vec{r}'|}\;\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)\,. \end{aligned} \tag{3.6}$$

We suppress the overline over the retarded Green function in this case. The vector and scalar potentials are given by

$$\begin{aligned} \Phi\left(\vec{r},t\right) &= \int \mathrm{d}^3r'\mathrm{d}t'\left[\rho\left(\vec{r}',t'\right)\right]\;G_R\left(\vec{r}-\vec{r}',t-t'\right) \\ &= \int \mathrm{d}^3r'\mathrm{d}t'\left[\rho\left(\vec{r}',t'\right)\right]\;\left\{\Theta\left(t-t'\right)\frac{1}{4\pi\epsilon_0|\vec{r}-\vec{r}'|}\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)\right\}\,, \end{aligned} \tag{3.7a}$$

$$\begin{aligned} c\vec{A}\left(\vec{r},t\right) &= \int \mathrm{d}^3r'\mathrm{d}t'\left[\frac{1}{c}\;\vec{J}\left(\vec{r}',t'\right)\right]\;G_R\left(\vec{r}-\vec{r}',t-t'\right) \\ &= \int \mathrm{d}^3r'\mathrm{d}t'\left[\frac{1}{c}\;\vec{J}\left(\vec{r}',t'\right)\right]\;\left\{\Theta\left(t-t'\right)\frac{1}{4\pi\epsilon_0|\vec{r}-\vec{r}'|}\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)\right\}\,. \end{aligned} \tag{3.7b}$$

Possible solutions $\Phi_{\mathrm{hom}}\left(\vec{r},t\right)$ and $c\vec{A}_{\mathrm{hom}}\left(\vec{r},t\right)$ of the homogeneous wave equation have been suppressed. Here, we have used

$$\delta\left(|\vec{r}-\vec{r}| - c\left(t-t'\right)\right) = \frac{1}{c}\;\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)\,. \tag{3.8}$$

After the t' integration, one finds

$$\Phi\left(\vec{r},t\right) = \Phi_{\mathrm{hom}}\left(\vec{r},t\right) + \int \mathrm{d}^3r'\left[\rho\left(\vec{r}',t-\frac{|\vec{r}-\vec{r}'|}{c}\right)\right]\;\frac{1}{4\pi\epsilon_0|\vec{r}-\vec{r}'|}\,, \tag{3.9a}$$

$$c\vec{A}\left(\vec{r},t\right) = c\vec{A}_{\mathrm{hom}}\left(\vec{r},t\right) + \int \mathrm{d}^3r'\left[\frac{1}{c}\vec{J}\left(\vec{r}',t-\frac{|\vec{r}-\vec{r}'|}{c}\right)\right]\;\frac{1}{4\pi\epsilon_0|\vec{r}-\vec{r}'|}\,. \tag{3.9b}$$

Here, $\vec{A}_{\mathrm{hom}}$ and Φ_{hom} are solutions to the homogenous wave equations (no sources); in many cases they will be taken to be zero. This concerns the solutions in coordinate space. Working in SI units, the latter equation can be rewritten as

$$\vec{A}\left(\vec{r},t\right) = \vec{A}_{\mathrm{hom}}\left(\vec{r},t\right) + \int \mathrm{d}^3r'\underbrace{\left[\frac{1}{\epsilon_0\;c^2}\right]}_{=\mu_0}\;\vec{J}\left(\vec{r}',t-\frac{|\vec{r}-\vec{r}'|}{c}\right)\;\frac{1}{4\pi|\vec{r}-\vec{r}'|}\,, \tag{3.10}$$

and we see that the vacuum permeability μ_0 naturally appears in the calculation.

We can introduce Fourier transforms in the two Eqs. (3.9a) and (3.9b) with respect to the time variables t and $t - |\vec{r} - \vec{r}'|/c$, directly, with the results

$$\int \frac{\mathrm{d}\omega}{2\pi} \widetilde{\Phi}\,(\vec{r},\omega)\,\mathrm{e}^{-\mathrm{i}\omega t} = \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}^3 r' \widetilde{\rho}\,(\vec{r}',\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\;\mathrm{e}^{\mathrm{i}\omega|\vec{r}-\vec{r}'|/c}\;\frac{1}{4\pi\epsilon_0|\vec{r}-\vec{r}'|}\,, \tag{3.11a}$$

$$\int \frac{\mathrm{d}\omega}{2\pi} c\widetilde{\vec{A}}\,(\vec{r},\omega)\,\mathrm{e}^{-\mathrm{i}\omega t} = \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}^3 r' \frac{1}{c} \widetilde{\vec{J}}\,(\vec{r}',\omega)\;\;\mathrm{e}^{-\mathrm{i}\omega t}\;\mathrm{e}^{\mathrm{i}\omega|\vec{r}-\vec{r}'|/c}\;\frac{1}{4\pi\epsilon_0|\vec{r}-\vec{r}'|}\,. \tag{3.11b}$$

We can now alternatively read off Fourier components with respect to the expression $\exp(-\mathrm{i}\omega\,t)$, or apply the integral operator $\int \mathrm{d}t\;\exp(\mathrm{i}\omega' t)$ to both sides of Eq. (3.11), integrate over t, and use the resulting Dirac δ function $\delta(\omega - \omega')$ in order to carry out the ω integral, and, as a last step, change back the notation according to $\omega' \to \omega$. In any case, one obtains

$$\widetilde{\Phi}\,(\vec{r},\omega) = \int \mathrm{d}^3 r'\;\widetilde{\rho}\,(\vec{r}',\omega)\;\frac{\mathrm{e}^{\mathrm{i}\omega|\vec{r}-\vec{r}'|/c}}{4\pi\epsilon_0\,|\vec{r}-\vec{r}'|} = \int \mathrm{d}^3 r'\;\widetilde{\rho}\,(\vec{r}',\omega)\;\widetilde{G}_R(\vec{r}-\vec{r}',\omega)\,, \tag{3.12a}$$

$$c\,\widetilde{\vec{A}}\,(\vec{r},\omega) = \int \mathrm{d}^3 r'\;\frac{1}{c}\,\widetilde{\vec{J}}\,(\vec{r}',\omega)\;\frac{\mathrm{e}^{\mathrm{i}\omega|\vec{r}-\vec{r}'|/c}}{4\pi\epsilon_0\,|\vec{r}-\vec{r}'|} = \int \mathrm{d}^3 r'\;\frac{1}{c}\,\widetilde{\vec{J}}\,(\vec{r}',\omega)\;\widetilde{G}_R(\vec{r}-\vec{r}',\omega)\,. \tag{3.12b}$$

The notation

Radiation/Retarded Green Function: $$\widetilde{G}_R(\vec{r}-\vec{r}',\omega) = \frac{\mathrm{e}^{\mathrm{i}\omega|\vec{r}-\vec{r}'|/c}}{4\pi\epsilon_0\,|\vec{r}-\vec{r}'|} \tag{3.13}$$

constitutes a Fourier transformation of the original space-time Green function with respect to time.

In electrostatics, we had encountered the Poisson equation

$$\Phi\,(\vec{r}) = -\frac{1}{\epsilon_0} \int g\,(\vec{r},\vec{r}')\;\rho\,(\vec{r}')\;\mathrm{d}^3 r'\,. \tag{3.14}$$

One possible solution for $g\,(\vec{r},\vec{r}')$, for vacuum boundary conditions, is

$$g\,(\vec{r},\vec{r}') = -\frac{1}{4\pi\,|\vec{r}-\vec{r}'|}\,, \qquad \vec{\nabla}^2 g\,(\vec{r},\vec{r}') = \delta^{(3)}\,(\vec{r}-\vec{r}')\;, \tag{3.15}$$

so that

$$\Phi\,(\vec{r}) = \int \mathrm{d}^3 r'\;\rho\,(\vec{r}')\;\frac{1}{4\pi\epsilon_0\,|\vec{r}-\vec{r}'|}\,. \tag{3.16}$$

Comparing Eq. (3.12a) to Eq. (3.16), we see that the Poisson equation is the limit of the "radiation generation equation" in the limit of low frequency, $\omega \to 0$, and that

Connection to Electrostatics: $$\widetilde{G}_R(\vec{r}-\vec{r}',\omega \longrightarrow 0) = -\frac{1}{\epsilon_0}\,g(\vec{r},\vec{r}')\,. \tag{3.17}$$

This result could have been obtained differently, starting from the Green function

$$G_R\,(\vec{r}-\vec{r}',t-t') = G_R\,(\Delta\vec{r},\Delta t) = \Theta\,(\Delta t)\;\frac{1}{4\pi\epsilon_0|\Delta\vec{r}|}\;\delta\left(\Delta t - \frac{|\Delta\vec{r}|}{c}\right)\,. \tag{3.18}$$

Its Fourier transform is

$$\widetilde{G}_R\,(\Delta\vec{r},\omega) = \int \mathrm{d}\tau\;\mathrm{e}^{\mathrm{i}\omega\tau}\;\Theta\,(\tau)\;\frac{1}{4\pi\epsilon_0|\Delta\vec{r}|}\;\delta\left(\tau - \frac{|\Delta\vec{r}|}{c}\right) = \frac{\mathrm{e}^{\mathrm{i}\omega|\Delta\vec{r}|/c}}{4\pi\epsilon_0\,|\Delta\vec{r}|} \tag{3.19}$$

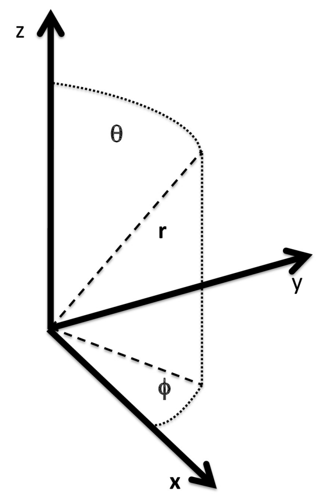


Figure 3.1: Illustration of the canonical definition of spherical coordinates.

which explains Eq. (3.13). This calculation identifies Eqs. (3.12a) and (3.12b) as the Fourier transforms of Eqs. (3.7a) and (3.7b), because convolution in time space means multiplication in frequency space. Moreover, as G_R fulfills the defining equation for the radiative Green function, given in Eq. (2.13),

Green Function in Coordinate Space: $$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right) G_R\left(\vec{r}-\vec{r}\,', t-t'\right) = \frac{1}{\epsilon_0}\,\delta^{(3)}\left(\vec{r}-\vec{r}\,'\right)\,\delta\left(t-t'\right)\,, \tag{3.20}$$

the Fourier transform $\widetilde{G}_R(\vec{r}-\vec{r}\,', \omega)$ fulfills the Helmholtz equation

$$\left(-\frac{\omega^2}{c^2} - \vec{\nabla}^2\right) \widetilde{G}_R\left(\vec{r}-\vec{r}\,', \omega\right) = \frac{1}{\epsilon_0}\,\delta\left(\vec{r}-\vec{r}\,'\right)\,. \tag{3.21}$$

Setting $\omega^2/c^2 = k^2$, we obtain the (inhomogeneous)

Helmholtz Equation: $$\left(\vec{\nabla}^2 + k^2\right)\,\widetilde{G}_R\left(\vec{r}-\vec{r}\,', \omega\right) = -\frac{1}{\epsilon_0}\,\delta\left(\vec{r}-\vec{r}\,'\right)\,. \tag{3.22}$$

If the right-hand side vanishes, the equation becomes homogeneous. We now study series expansions for the solution of the homogeneous equation, from which, by our experience gained in connection with electrostatics, solutions to the inhomogeneous problem can be obtained.

3.2.2 Helmholtz Equation in Spherical Coordinates

Eventually, we will attempt to calculate the representation of the Helmholtz Green function in spherical coordinates. To this end, we search for solutions to the homogeneous equation first. Then, we use the representation of the Dirac-δ in spherical coordinates in order to write the equations for the angular momentum components. In the last step, we match the regular and irregular solutions at the cusp, by integration over the point where $r = r'$, in an infinitesimal interval. The homogeneous solutions will be written in terms of spherical harmonics. We shall describe the homogeneous solutions to the Helmholtz equation first; this requires a discussion of the spherical Bessel functions. So, a brief indication of the solutions of the Helmholtz equation in spherical coordinates (r, θ, φ) complements our discussion (see also Fig. 3.1). The homogeneous

Helmholtz equation reads

$$\left(\vec{\nabla}^2 + k^2\right) \Phi(r,\theta,\varphi) = \left(\frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + k^2\right)\Phi(r,\theta,\varphi)$$
$$= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\vec{L}^2}{r^2} + k^2\right)\Phi(r,\theta,\varphi) = 0\,, \tag{3.23}$$

with $|m| \leq \ell = 0, 1, 2, \ldots$. The solution reads

$$\Phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \left[a_{\ell,m}\, j_\ell(kr) + b_{\ell,m}\, y_\ell(kr)\right] Y_{\ell,m}(\theta,\varphi)\,, \tag{3.24}$$

where the separation constants $a_{\ell,m}$ and $b_{\ell,m}$ can take arbitrary values, and each function in the set fulfills the Helmholtz equation, separately. The spherical Bessel and Neumann functions fulfill for $\ell = 0, 1, 2, \ldots$,

$$j_\ell(x) = \left(\frac{\pi}{2x}\right)^{1/2} J_{\ell+1/2}(x)\,, \qquad y_\ell(x) = \left(\frac{\pi}{2x}\right)^{1/2} Y_{\ell+1/2}(x)\,. \tag{3.25}$$

Special values and asymptotics are given as follows,

$$j_0(x) = \frac{\sin x}{x}\,, \qquad j_\ell(x) \stackrel{x\to 0}{\to} \frac{x^\ell}{(2\ell+1)!!}\,, \qquad j_\ell(x) \stackrel{x\to\infty}{\to} \frac{1}{x}\sin\left(x - \frac{\ell\pi}{2}\right)\,, \tag{3.26a}$$

$$y_0(x) = -\frac{\cos x}{x}\,, \qquad y_\ell(x) \stackrel{x\to 0}{\to} -\frac{(2\ell+1)!!}{x^{\ell+1}}\,, \qquad y_\ell(x) \stackrel{x\to\infty}{\to} -\frac{1}{x}\cos\left(x - \frac{\ell\pi}{2}\right)\,. \tag{3.26b}$$

Because the Bessel functions have to evaluated for half-integer index, we should indicate valid integral representations. Most integral representations are not valid in the entire complex plane, and we shall indicate two representations which are valid For $|\arg(z)| < \pi/2$, i.e. $\mathrm{Re}\, z > 0$. Indeed, according to Eq. (10.9.6) of [F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, NY (1974)], we have

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos\left[z\,\sin(\theta) - \nu\,\theta\right]\,\mathrm{d}\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty \exp[-z\,\sinh(\theta) - \nu\,\theta]\,\mathrm{d}\theta\,. \tag{3.27}$$

Furthermore, according to Eq. (10.9.7) of the same work of F. W. J. Olver, we have

$$Y_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin\left[z\,\sin(\theta) - \nu\,\theta\right]\,\mathrm{d}\theta - \frac{1}{\pi}\int_0^\infty \left\{\mathrm{e}^{\nu\,t} + \mathrm{e}^{-\nu\,t}\cos(\nu\,\pi)\right\}\,\mathrm{e}^{-z\,\sinh(t)}\,\mathrm{d}\theta\,. \tag{3.28}$$

For integer $\nu = n$ with integer $n \in \mathbb{N}_0$, the second term on the right-hand side of Eqs. (3.27) vanishes.

For completeness, we should also indicate Schläfli's contour integral representation of the Bessel function,

$$J_\nu(z) = \frac{1}{2\,\pi\,\mathrm{i}}\oint_\infty^{0^+} \exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right]\, t^{-\nu-1}\,\mathrm{d}t\,, \tag{3.29}$$

where 0^+ indicates that the origin is to be encircled in the counterclockwise (mathematically positive) direction. There is also a connection of the Bessel and Neuman functions valid for non-integer order ν,

$$Y_\nu(z) = \frac{J_\nu(z)\,\cos(\nu\,\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}\,. \tag{3.30}$$

The spherical Bessel and Neumann functions are obtained for half-integer $\nu = \ell + 1/2$.

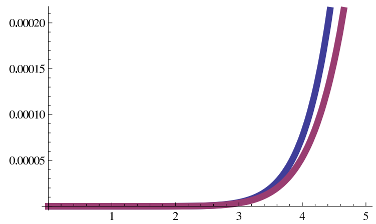


Figure 3.2: Verification of Eq. (3.35).

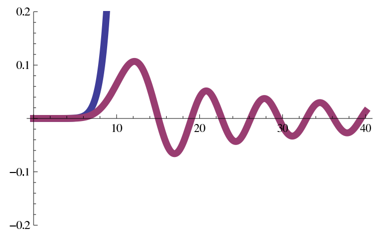


Figure 3.3: Verification of Eq. (3.35) on a larger scale.

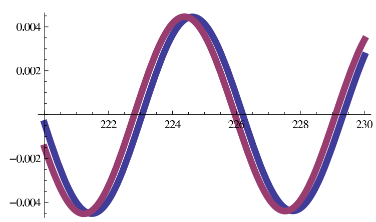


Figure 3.4: Verification of Eq. (3.36).

Boundary conditions on the solution restrict the available values for the separation constants, and this phenomenon will now be studied in more detail. In order for Eq. (3.24) to represent a general solution to the Helmholtz equation, we have to demand that

$$\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{\ell(\ell+1)}{r^2}+k^2\right)j_\ell(k\,r)=0\,. \tag{3.31}$$

A basic relation is that the function $z=x^\alpha\,J_n(\beta\,x^\gamma)$ fulfills the Bessel differential equation

$$\frac{\partial^2 z}{\partial x^2}-\frac{2\alpha-1}{x}\frac{\partial z}{\partial x}+\left(\beta^2\gamma^2x^{2\gamma-2}+\frac{\alpha^2-n^2\gamma^2}{x^2}\right)z=0\,. \tag{3.32}$$

For our case, we set $\gamma=1$, $\alpha=-\frac{1}{2}$, $\beta=k$, and $n=\ell+1/2$. Then,

$$\frac{\partial^2 z}{\partial x^2}-\frac{-2}{x}\frac{\partial z}{\partial x}+\left(k^2+\frac{\frac{1}{4}-(\ell+\frac{1}{2})^2}{x^2}\right)z=0\quad\Rightarrow\quad\frac{\partial^2 z}{\partial x^2}+\frac{2}{x}\frac{\partial z}{\partial x}+\left(k^2-\frac{\ell(\ell+1)}{x^2}\right)z=0\,, \tag{3.33}$$

which is just the desired Eq. (3.31). The complementary solution y_ℓ also fulfills the defining equation. Let us verify the asymptotic formulas

$$j_\ell(x)\stackrel{x\to 0}{\to}\frac{x^\ell}{(2\ell+1)!!}\,,\qquad j_\ell(x)\stackrel{x\to\infty}{\to}\frac{1}{x}\,\sin\left(x-\frac{\ell\pi}{2}\right)\,. \tag{3.34}$$

by way of example. For $\ell = 10$, we have

$$j_{\ell=10}(x) \overset{x\to 0}{\propto} \frac{x^{10}}{13\,749\,310\,575}\,. \tag{3.35}$$

This is confirmed in Fig. 3.2. On a larger scale, there are deviations (see Fig. 3.3). For large argument,

$$j_{\ell=10}(x) \overset{x\to\infty}{\propto} \frac{1}{x}\,\sin\left(x - \frac{10\,\pi}{2}\right)\,. \tag{3.36}$$

This is verified in Fig. 3.4. The first few spherical Bessel functions are

$$j_0(z) = \frac{\sin z}{z}\,, \qquad j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}\,, \tag{3.37a}$$

$$j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right)\,\sin z - \frac{3}{z^2}\,\cos z\,, \tag{3.37b}$$

and

$$y_0(z) = -\frac{\cos z}{z}\,, \qquad y_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z}\,, \tag{3.38a}$$

$$y_2(z) = \left(-\frac{3}{z^3} + \frac{1}{z}\right)\,\cos z - \frac{3}{z^2}\,\sin z\,. \tag{3.38b}$$

The eigenfunctions satisfy the following orthogonality condition,

$$\langle \psi_{lm} | \psi_{\ell' m'} \rangle = \int_0^\infty \mathrm{d}r\, r^2\, j_\ell\,(k\,r)\; j_{\ell'}\,(k'\,r)\; \int \mathrm{d}\Omega\; Y_{\ell m}\,(\theta,\varphi)^*\;\; Y_{\ell' m'}\,(\theta,\varphi) = \frac{\pi}{2\,k\,k'}\,\delta\,(k-k')\;\delta_{\ell\ell'}\;\delta_{mm'}\,. \tag{3.39}$$

Remark: The radial part of the Laplacian operator is given in Eq. (3.23) is we set $\vec{L}^2 = 0$. If we assume that f is a radially symmetric function, $f = f(|\vec{r}|) = f(r)$, then the following three forms of the Laplacian are equivalent,

$$\vec{\nabla}^2 f(r) = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\, f(r) = \left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\right)\, f(r) = \frac{1}{r}\frac{\partial^2}{\partial r^2}\,(r\,f(r)) \tag{3.40}$$

The first of these is the "standard form" (in d dimensions, replace $2/r \to (d-1)/r$. The second is useful for integrating over the cusp of the Helmholtz Green function, as discussed below, and the third is useful for calculations with the radial component of the Schrödinger equation of the hydrogen atom.

3.3 Localized Harmonically Oscillating Sources

3.3.1 Basic Formulas

We consider first sources which oscillate at a fixed frequency ω,

$$\rho\,(\vec{r},t) = \rho_0\;\,(\vec{r})\;\exp\,(-\mathrm{i}\omega t) \qquad \vec{J}\,(\vec{r},t) = \vec{J}_0\,(\vec{r})\;\exp\,(-\mathrm{i}\omega t)\;. \tag{3.41}$$

If we now assume that the vector potentials and fields have the same harmonic dependence,

$$\begin{aligned}\vec{A}\,(\vec{r},t) &= \vec{A}_0\,(\vec{r})\;\exp\,(-\mathrm{i}\omega t)\;,\\ \vec{B}\,(\vec{r},t) &= \vec{B}_0\,(\vec{r})\;\exp\,(-\mathrm{i}\omega t)\;,\\ \vec{E}\,(\vec{r},t) &= \vec{E}_0\,(\vec{r})\;\exp\,(-\mathrm{i}\omega t)\;,\end{aligned} \tag{3.42}$$

then for $X = \rho, \vec{J}, \vec{A}, \vec{B}, \vec{E}$, we have in Fourier space,

$$X(\vec{r}, \omega') = X_0(\vec{r}) \left[2\pi\delta(\omega - \omega')\right] . \tag{3.43}$$

We recall that Eq. (3.12b) reads

$$\tilde{\vec{A}}(\vec{r}, \omega') = \int \mathrm{d}^3 r' \, \frac{1}{c^2} \, \tilde{\vec{J}}(\vec{r}', \omega') \, \frac{\mathrm{e}^{\mathrm{i}\omega' |\vec{r} - \vec{r}'|/c}}{4\pi\epsilon_0 \, |\vec{r} - \vec{r}'|} = \int \mathrm{d}^3 r' \, \tilde{\vec{J}}(\vec{r}', \omega') \, \frac{\exp\left(\mathrm{i}\omega' \, |\vec{r} - \vec{r}'|/c\right)}{4\pi\epsilon_0 c^2 \, |\vec{r} - \vec{r}'|} . \tag{3.44}$$

Using Eq. (3.43), we can reformulate this as

$$\tilde{\vec{A}}_0(\vec{r}) \left[2\pi\delta(\omega - \omega')\right] = \int \mathrm{d}^3 r' \, \tilde{\vec{J}}_0(\vec{r}') \, \left[2\pi\delta(\omega - \omega')\right] \frac{\exp\left(\mathrm{i}\omega' \, |\vec{r} - \vec{r}'|/c\right)}{4\pi\epsilon_0 c^2 \, |\vec{r} - \vec{r}'|} . \tag{3.45}$$

After an integration over $\int \mathrm{d}\omega'$, we obtain

$$\vec{A}_0(\vec{r}) = \int \mathrm{d}^3 r' \vec{J}_0(\vec{r}') \frac{\exp\left(\mathrm{i}\omega |\vec{r} - \vec{r}'|/c\right)}{4\pi\epsilon_0 c^2 \, |\vec{r} - \vec{r}'|} , \qquad \vec{A}(\vec{r}, t) = \vec{A}_0(\vec{r}) \, \exp\left(-\mathrm{i}\omega t\right) . \tag{3.46}$$

The spatial dependence of the magnetic induction is given by

$$\vec{B}_0(\vec{r}) = \vec{\nabla} \times \vec{A}_0(\vec{r}) . \tag{3.47}$$

We assume that the radiated fields are calculated at the space-time point $(\vec{r}, t)$, and that the oscillating charge distribution is concentrated in a small spatial region of locations about $\vec{r}'$, far away from the observation point $\vec{r}$. Then, at the observation point, we have $\vec{J}(\vec{r}, t) = 0$, and the electric field is given by

$$\begin{aligned} \vec{\nabla} \times \vec{B}(\vec{r}, t) &= \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}(\vec{r}, t) = -\mathrm{i} \frac{\omega}{c^2} \, \vec{E}(\vec{r}, t) , \\ \vec{E}(\vec{r}, t) &= \frac{\mathrm{i}c^2}{\omega} \left(\vec{\nabla} \times \vec{B}(\vec{r}, t)\right) , \qquad \vec{E}_0(\vec{r}) = \frac{\mathrm{i}c^2}{\omega} \left(\vec{\nabla} \times \vec{B}_0(\vec{r})\right) . \end{aligned} \tag{3.48}$$

We thus do not even need to calculate the scalar potential, because both the magnetic as well as the electric field are given in terms of the vector potential,

$$\vec{B}_0(\vec{r}) = \vec{\nabla} \times \vec{A}_0(\vec{r}) , \qquad \vec{E}_0(\vec{r}) = \frac{\mathrm{i}c^2}{\omega} \, \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}_0(\vec{r})\right) . \tag{3.49}$$

We have already restricted our sources to be localized with a characteristic dimension, d, satisfying

$$d \ll \frac{c}{\omega} = \frac{\lambda}{2\pi} . \tag{3.50}$$

The exponential term in the integrand for the vector potential suggests that the potential will have a different spatial dependence depending on the range of $\vec{r}$:

$$d \ll r \ll \frac{c}{\omega} \qquad \text{(near field or static zone, less than a wavelength away)} \tag{3.51}$$

$$r \gg \frac{c}{\omega} \qquad \text{(far field or radiation zone, many wavelengths away).} \tag{3.52}$$

The retarded Green function fulfills

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right) G_R(\vec{r} - \vec{r}', t - t') = \frac{1}{\epsilon_0} \, \delta(\vec{r} - \vec{r}') \, \delta(t - t') . \tag{3.53}$$

We had previously derived that

$$\widetilde{G}_R(\vec{r}-\vec{r}',\omega) = \int_{-\infty}^{\infty} \mathrm{d}\tau\, G_R(\vec{r}-\vec{r}',\tau)\exp(\mathrm{i}\,\omega\,\tau) = \frac{1}{4\pi\epsilon_0}\,\frac{\exp(\mathrm{i}\omega|\vec{r}-\vec{r}'|/c)}{|\vec{r}-\vec{r}'|}\,. \tag{3.54}$$

Consequently, the Fourier transform with respect to time/frequency fulfills,

$$\left(-\frac{\omega^2}{c^2}-\vec{\nabla}^2\right)\widetilde{G}_R(\vec{r}-\vec{r}',\omega) = \frac{1}{\epsilon_0}\,\delta(\vec{r}-\vec{r}')\,, \tag{3.55}$$

or with $k=\omega/c$,

$$\left(\vec{\nabla}^2+k^2\right)\,\widetilde{G}_R(\vec{r}-\vec{r}',\omega) = -\frac{1}{\epsilon_0}\,\delta(\vec{r}-\vec{r}')\,. \tag{3.56}$$

Let us define

$$G_0(\vec{r}-\vec{r}',k) \equiv \widetilde{G}_R(\vec{r}-\vec{r}',\omega = c\,k)\,. \tag{3.57}$$

Using $\vec{\nabla}r = \vec{\nabla}|\vec{r}| = \vec{r}/r$, we can convince ourselves that G_0 really fulfills the defining equation of the Helmholtz Green function,

$$\begin{aligned}
\vec{\nabla}^2\frac{\exp(\mathrm{i}k\,r)}{r} &= \vec{\nabla}\cdot\vec{\nabla}\frac{\exp(\mathrm{i}k\,|\vec{r}|)}{r} = \vec{\nabla}\cdot\left(\exp(\mathrm{i}k\,|\vec{r}|)\,\left[\frac{\mathrm{i}k}{r}\vec{\nabla}r+\vec{\nabla}\frac{1}{r}\right]\right)\,,\\
&= \mathrm{e}^{\mathrm{i}kr}\left(-\frac{\vec{k}^2}{r}\vec{\nabla}r\cdot\vec{\nabla}r+\mathrm{i}k\,\vec{\nabla}r\cdot\vec{\nabla}\left(\frac{1}{r}\right)+\frac{\mathrm{i}k}{r}\vec{\nabla}\cdot\vec{\nabla}r+\mathrm{i}k\vec{\nabla}r\cdot\vec{\nabla}\frac{1}{r}+\vec{\nabla}^2\frac{1}{r}\right)\\
&= \mathrm{e}^{\mathrm{i}kr}\left(-\frac{\vec{k}^2}{r}\left(\frac{\vec{r}}{r}\cdot\frac{\vec{r}}{r}\right)+\mathrm{i}k\,\frac{\vec{r}}{r}\cdot\left(\frac{-\vec{r}}{r^3}\right)+\frac{\mathrm{i}k}{r}\left(\frac{2}{r}\right)+\mathrm{i}k\left(\frac{\vec{r}}{r}\right)\cdot\left(\frac{-\vec{r}}{r^3}\right)+\vec{\nabla}^2\frac{1}{r}\right)\\
&= \mathrm{e}^{\mathrm{i}kr}\left(\frac{-k^2}{r}-\frac{\mathrm{i}k}{r^2}+\frac{2\mathrm{i}k}{r^2}-\frac{\mathrm{i}k}{r^2}+\vec{\nabla}^2\frac{1}{r}\right)\\
&= \mathrm{e}^{\mathrm{i}kr}\left(\frac{-k^2}{r}+\vec{\nabla}^2\frac{1}{r}\right)\,,
\end{aligned}$$

$$\Rightarrow\ \left(\vec{\nabla}^2+k^2\right)\,G_0(\vec{r},k) = \left(\vec{\nabla}^2+k^2\right)\,\frac{\exp\left(\mathrm{i}|\vec{k}||\vec{r}|\right)}{4\pi\epsilon_0\,r} = -\frac{1}{\epsilon_0}\delta(\vec{r})\,\exp(\mathrm{i}k\,r) = -\frac{1}{\epsilon_0}\delta(\vec{r})\,. \tag{3.58}$$

The homogeneous Helmholtz equation is normally given as the Fourier transform (with respect to time) of the wave equation,

Wave Equation with Sources:
$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\vec{\nabla}^2\right)\psi(\vec{r},t) = \frac{1}{\epsilon_0}\,F(\vec{r},t)\,, \tag{3.59}$$

such that for harmonic oscillations at frequency ω,

Helmholtz Equation with Sources:
$$\left(\frac{\omega^2}{c^2}+\vec{\nabla}^2\right)\psi(\vec{r}) = -\frac{1}{\epsilon_0}\,F(\vec{r})\,, \tag{3.60}$$

the vector potential is given as

$$\vec{A}_0(\vec{r}) = \int \mathrm{d}^3r'\,\vec{J}_0(\vec{r}')\,\frac{\exp(\mathrm{i}\omega|\vec{r}-\vec{r}'|/c)}{4\pi\epsilon_0\,c^2\,|\vec{r}-\vec{r}'|}\,. \tag{3.61}$$

The Green function for the Helmholtz equation fulfills, according to the above considerations,

$$\begin{aligned}
\left[\vec{\nabla}^2+k^2\right]\,G_0(\vec{r}-\vec{r}',k) &= -\frac{1}{\epsilon_0}\delta(\vec{r}-\vec{r}')\,,\\
G_0(\vec{r}-\vec{r}',k) &= \frac{\exp(\mathrm{i}\,k\,|\vec{r}-\vec{r}'|)}{4\pi\epsilon_0\,|\vec{r}-\vec{r}'|}\,,\qquad k\equiv|\vec{k}| = \frac{\omega}{c}\,.
\end{aligned} \tag{3.62}$$

In order to evaluate $\vec{A}_0\,(\vec{r})$, one needs an appropriate expansion for the Green function, which in spherical coordinates reads as follows,

$$G_0(\vec{r}-\vec{r}',k) = \sum_{\ell=0}^{\infty} \sum_{m=-l}^{\ell} g_\ell\,(r,r')\;\; Y_{\ell m}\,(\theta,\varphi)\;\; Y_{\ell m}\,(\theta',\varphi')^* \;, \tag{3.63}$$

where we have $g_\ell\,(r,r') = g_\ell\,(k,r,r')$ in terms of a functional dependence. The dependence on k is suppressed in our notation. The action of the Laplacian operator onto a test function of the form $R(r)\,Y_{\ell m}(\hat{r})$ is as follows,

$$\left(\vec{\nabla}^2 + \vec{k}^2\right)\; [R(r)\,Y_{\ell m}(\theta,\varphi)] = \left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + k^2 - \frac{\ell\,(\ell+1)}{r^2}\right)\; [R(r)\,Y_{\ell m}(\theta,\varphi)]\;, \tag{3.64a}$$

$$\delta^{(3)}(\vec{r}-\vec{r}') = \frac{1}{r^2\,\sin\theta}\,\delta(r-r')\,\delta(\theta-\theta')\,\delta(\varphi-\varphi')\,, \tag{3.64b}$$

$$\sum_{\ell m} Y_{\ell m}(\theta,\varphi)\,Y^*_{\ell m}(\theta',\varphi') = \frac{1}{\sin\theta}\,\delta(\theta-\theta')\,\delta(\varphi-\varphi')\,. \tag{3.64c}$$

If, independent of ℓ, $g_\ell\,(r,r')$ fulfills

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\; g_\ell\,(r,r')\right) + \left(k^2\; r^2 - \ell\;(\ell+1)\right) g_\ell\,(r,r') = -\frac{1}{\epsilon_0}\,\delta\,(r-r')\;, \tag{3.65}$$

then G_0 will be a suitable Green function of the Helmholtz equation. The function $g_\ell\,(r,r')$ can be expanded in solutions to the spherical Bessel equation,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}}{\mathrm{d}r} f_\ell\,(k\,r)\right) + \left(k^2\,r^2 - \ell\,(\ell+1)\right)\, f_\ell\,(k\,r) = 0\,, \tag{3.66}$$

where the general solution, $f_\ell\,(kr)$, is a linear combination of the two spherical Bessel functions,

$$f_\ell\,(kr) = a_\ell\, j_\ell(kr) + b_\ell\, y_\ell(kr)\,. \tag{3.67}$$

We recall that the defining differential equation for spherical Bessel functions is given as

$$\left(\frac{\partial^2}{\partial z^2} + \frac{2}{z}\,\frac{\partial}{\partial z} - \frac{\nu\,(\nu+1)}{z^2} + 1\right)\; j_\nu(z) = 0\,. \tag{3.68}$$

The recursion relations for spherical Bessel functions are given as follows,

$$j_{\nu-1}(z) + j_{\nu+1}(z) = \frac{2\nu+1}{z}\, j_\nu(z)\,, \tag{3.69a}$$

$$\nu\, j_{\nu-1}(z) - (\nu+1)\; j_{\nu+1}(z) = (2\nu+1)\, j'_\nu(z)\,, \tag{3.69b}$$

We also recall that one can express the derivatives as follows,

$$j'_\nu(z) = j_{\nu-1}(z) - \frac{\nu+1}{z}\, j_\nu(z)\,, \tag{3.70a}$$

$$j'_\nu(z) = \,-\,j_{\nu+1}(z) + \frac{\nu}{z}\, j_\nu(z)\,. \tag{3.70b}$$

The $j_\ell(kr)$ are finite at $r=0$, whereas the $y_\ell(kr)$ diverge,

$$j_\ell\,(z) \sim \frac{z^\ell}{(2\ell+1)!!}\,, \qquad z\to 0\,, \qquad y_\ell\,(z) \sim -\frac{(2\ell-1)!!}{z^{\ell+1}}\,, \qquad z\to 0\,, \tag{3.71a}$$

$$j_\ell\,(z) \sim \frac{1}{z}\,\sin\left(z - \frac{\ell\pi}{2}\right)\,, \qquad z\to\infty\,, \qquad y_\ell\,(z) \sim -\frac{1}{z}\,\cos\left(z - \frac{\ell\pi}{2}\right)\,, \qquad z \gg 1\,. \tag{3.71b}$$

The double factorial is explained as

$$(2\ell+1)!! = (2\ell+1)(2\ell-1)(2\ell-3)\ldots 5\cdot 3\cdot 1\,. \tag{3.72}$$

Since the Green's function is regular at $r = 0$, and in particular, for $r < r'$, we must choose the regular solution, i.e., the spherical Bessel function j_ℓ, in this domain,

$$g_\ell\,(r,r') = a\,(r')\,j_\ell\,(k\,r)\ , \qquad r < r'\,. \tag{3.73}$$

For $r > r'$, we certainly need a contribution of the Bessel y_ℓ which is not regular at the origin. However, if we wish to construct a Wronskian upon action of the radial differential operator, then we need to add a contribution from j_ℓ as well. A natural assumption is to make an ansatz for the solution to be an incoming wave or an outgoing wave,

$$\propto \exp\left[\mathrm{i}\left(kr - \frac{\ell\pi}{2}\right)\right] \ \text{(outgoing)}\,, \qquad \propto \exp\left[-\mathrm{i}\left(kr - \frac{\ell\pi}{2}\right)\right] \ \text{(incoming)}\,. \tag{3.74}$$

Two considerations support the outgoing wave.

(i) We assume that $r < r'$. The propagator propagates the point $\vec{r}'$ to $\vec{r}$. An incoming wave converges to the point $\vec{r}'$; it should be propagated to point $\vec{r}$. That means that since $\vec{r}'$ is the incoming point, in an eigenfunction decomposition of the Green function we would use the complex conjugate of the wave function at the incoming point. The outgoing wave is thus preferred because it is the complex conjugate of the incoming wave.

(ii) We can also consider the case $r > r'$, and take into account the fact that in this case, the wave is outgoing from the point r. Since we are contructing an outgoing wave, the ansatz with the outgoing wave again is preferred.

From the analogy with the Green function for electrostatics, we also conjecture that the solution should fall off as r^{-1} for large r. A possible approach then is to assume that

$$g_\ell\,(r,r') = b\,(r')\ \left[\mathrm{i}\,j_\ell\,(k\,r) - y_\ell\,(k\,r)\right]\ , \qquad r > r'\,. \tag{3.75}$$

Then, for $kr \gg 1$,

$$g_\ell\,(r,r') = b\,(r')\ \left[\mathrm{i}\,j_\ell\,(k\,r) - y_\ell\,(kr)\right] \longrightarrow b\,(r')\ \frac{1}{k\,r}\ \exp\left[\mathrm{i}\left(k\,r - \frac{\ell\pi}{2}\right)\right]\,. \tag{3.76}$$

Continuity at $r = r'$ is automatically fulfilled if we write the radial component of the Green functions as a product, supplementing for $a(r')$ and $b(r')$ the respective other solution of the radial Helmholtz equation,

$$g_\ell\,(r,r') = a_0\ j_\ell\,(k\,r_<)\ \left[\mathrm{i}\,j_\ell\,(k\,r_>) - y_\ell\,(k\,r_>)\right]\,. \tag{3.77}$$

Only the regular solution can be sustained at the lower argument, i.e., for $j_\ell\,(k\,r_<)$. In order to convince ourselves of the continuity, we consider r' to be constant and vary r, starting at $r = 0$. As we increase r, the variable r assumes the role of $r_<$ until $r = r'$ (cusp). At the cusp, it does not matter which identification we make, because r and r' are equal. For $r > r'$, the identification of $r_<$ and $r_>$ is different, but we have gone through the cusp where the two arguments were equal. In some sense, the cusp thus ensures the continuity.

The constant a_0 is determined from the discontinuity in the derivative of $g_\ell\,(r,r')$ at $r = r'$. Note that $g_\ell\,(r,r')$ is continuous at $r = r'$, but the derivative $\frac{\mathrm{d}}{\mathrm{d}r} g_\ell\,(r,r')$ is discontinuous. Only the region

$$r' - \epsilon \le r \le r' + \epsilon \tag{3.78}$$

gives a non-zero contribution. Taking notice of the continuity of the wave and of the discontinuity of its derivative at the cusp, which generates a Dirac-δ upon second differentiation, we have

$$\lim_{\epsilon\longrightarrow 0}\int\limits_{r'-\epsilon}^{r'+\epsilon}\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\,g_\ell\left(r,r'\right)\right)+\left(k^2\,r^2-\ell\left(\ell+1\right)\right)g_\ell((r,r')\right]\,\mathrm{d}r=\,-\,\frac{1}{\epsilon_0}\,\int_{r'-\epsilon}^{r'+\epsilon}\delta\left(r-r'\right)\,\mathrm{d}r\,,$$

$$\Rightarrow\qquad r^2\,\frac{\mathrm{d}}{\mathrm{d}r}g_\ell\left(r,r'\right)\Big|_{r'-\epsilon}^{r'+\epsilon}+0=\,-\,\frac{1}{\epsilon_0}\,. \tag{3.79}$$

The integrals of functions without a singularity vanishes when integrated over an infinitesimal integration interval. We can now identify the places where the differentiations take place, immediately to the right and left of the cusp,

$$r^2\,\frac{\mathrm{d}}{\mathrm{d}r}a_0\,j_\ell\left(k\,r_<\right)\,\left[\mathrm{i}\,j_\ell\left(k\,r_>\right)-y_\ell\left(k\,r_>\right)\right]\Big|_{r=r'-\epsilon}^{r=r'+\epsilon}=\,-\,\frac{1}{\epsilon_0}\,,$$

$$\Rightarrow\qquad r^2\,a_0\,j_\ell\left(k\,r'\right)\,\left(\frac{\mathrm{d}}{\mathrm{d}r}\left[\mathrm{i}j_\ell\left(k\,r\right)-y_\ell\left(k\,r\right)\right]\Big|_{r=r'+\epsilon}\right)$$

$$-r^2\,\left(\frac{\mathrm{d}}{\mathrm{d}r}a_0\,j_\ell\left(kr\right)\right)\Big|_{r=r'-\epsilon}\,\left[\mathrm{i}\,j_\ell\left(k\,r'\right)-y_\ell\left(k\,r'\right)\right]=\,-\,\frac{1}{\epsilon_0}$$

$$\Rightarrow\qquad r^2\,a_0\,\left(j_\ell\left(k\,r'\right)\,\frac{\mathrm{d}}{\mathrm{d}r'}\left[\mathrm{i}j_\ell\left(k\,r'\right)-y_\ell\left(k\,r'\right)\right]-\left(\frac{\mathrm{d}}{\mathrm{d}r'}j_\ell\left(kr'\right)\right)\,\left[\mathrm{i}\,j_\ell\left(k\,r'\right)-y_\ell\left(k\,r'\right)\right]\right)=\,-\,\frac{1}{\epsilon_0}\,. \tag{3.80}$$

So, as the term proportional to $j_\ell\,j_\ell'$ cancels, we have, replacing $r'\rightarrow r$,

$$a_0\,r^2\,\left[-j_\ell\left(k\,r\right)\,\frac{\mathrm{d}}{\mathrm{d}r}y_\ell\left(k\,r\right)+y_\ell\left(kr\right)\,\frac{\mathrm{d}}{\mathrm{d}r}j_\ell\left(k\,r\right)\right]=-\frac{1}{\epsilon_0}\,. \tag{3.81}$$

In the exercises, it will be shown that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\left(-j_\ell\left(k\,r\right)\,\frac{\mathrm{d}}{\mathrm{d}r}y_\ell\left(k\,r\right)+y_\ell\left(kr\right)\,\frac{\mathrm{d}}{\mathrm{d}r}j_\ell\left(k\,r\right)\right)\right]=0\,, \tag{3.82}$$

and so the Wronskian can be evaluated for any r. In particular, we can use the form for the two spherical Bessel functions as $r'\rightarrow 0$, according to Eq. (3.26),

$$a_0\,r^2\,\left\{\frac{-\left(k\,r\right)^\ell}{(2\ell+1)!!}\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(-\frac{(2\ell-1)!!}{\left(k\,r\right)^{l+1}}\right)\right]+\left(-\frac{(2\ell-1)!!}{\left(k\,r\right)^{\ell+1}}\right)\,\left[\frac{\mathrm{d}}{\mathrm{d}r}\frac{\left(k\,r\right)^\ell}{(2\ell+1)!!}\right]\right\}=\,-\,\frac{1}{\epsilon_0}\,,$$

$$a_0r^2\,\left[\frac{\left(k\,r\right)^\ell}{(2\ell+1)}\,\left(\frac{-(\ell+1)\,k}{\left(k\,r\right)^{\ell+2}}\right)+\left(-\frac{1}{\left(k\,r\right)^{\ell+1}}\right)\,\left(\frac{\ell k\,\left(k\,r\right)^{\ell-1}}{(2\ell+1)}\right)\right]=\,-\,\frac{1}{\epsilon_0}\,,$$

$$a_0\,r^2\,\left[\frac{-(\ell+1)k-\ell\,k}{(k\,r)^2\,\left(2\ell+1\right)}\right]=\,-\,\frac{1}{\epsilon_0}\,, \tag{3.83}$$

Finally,

$$a_0\left(\frac{-1}{k}\right)=-\frac{1}{\epsilon_0}\,,\qquad\qquad a_0=\frac{k}{\epsilon_0}\,. \tag{3.84}$$

Thus, the final form for $g_\ell\left(r,r'\right)$ is

$$\begin{aligned} g_\ell\left(r,r'\right)&=\frac{1}{\epsilon_0}\,k\,j_\ell\left(k\,r_<\right)\,\left[\mathrm{i}\,j_\ell\left(k\,r_>\right)-y_\ell\left(k\,r_>\right)\right]\\ &=\frac{1}{\epsilon_0}\,\mathrm{i}\,k\,j_\ell\left(k\,r_<\right)\,h_\ell^{(1)}\left(k\,r_>\right)\,, \end{aligned} \tag{3.85}$$

where

$$
\begin{aligned}
h_\ell^{(1)}(\rho) &= j_\ell(\rho) + \mathrm{i}\, y_\ell(\rho)\,, \\
h_\ell^{(2)}(\rho) &= j_\ell(\rho) - \mathrm{i}\, y_\ell(\rho)\,,
\end{aligned} \tag{3.86}
$$

are the Hankel functions. In spherical coordinates, the Green function thus becomes

Angular Momentum Decomposition of the Radiation Green Function:

$$
G_0(\vec{r}-\vec{r}\,',k) = \frac{\exp\left(\mathrm{i}\,k\,|\vec{r}-\vec{r}\,'|\right)}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|} = \frac{1}{\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \mathrm{i}\,k\,j_\ell(k\,r_<)\,h_\ell^{(1)}(k\,r_>)\,Y_{\ell m}(\theta,\varphi)\,Y_{\ell m}^*(\theta',\varphi')\,. \tag{3.87}
$$

In Physics/411, we had encoutered the

Angular Momentum Decomposition of the Green Function of Electrostatics: (3.88)

$$
-\frac{1}{\epsilon_0}g(\vec{r},\vec{r}\,') = \frac{1}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|} = \frac{1}{\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}}\,Y_{\ell m}(\theta,\varphi)\,Y_{\ell m}^*(\theta',\varphi')\,, \tag{3.89}
$$

based on the

Multipole Expansion:

$$
\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}}\,Y_{\ell m}(\theta,\varphi)\,Y_{\ell m}^*(\theta',\varphi')\,, \tag{3.90}
$$

which is the basis for all so called multipole expansions in electrodynamics, and thus very important. Comparing with Eq. (3.17), the matching is successful if

Matching with Electrostatics:

$$
G_0(\vec{r}-\vec{r}\,',k) \stackrel{k\to 0}{=} -\frac{1}{\epsilon_0}g(\vec{r},\vec{r}\,')\,, \qquad \mathrm{i}\,k\,j_\ell(k\,r_<)\,h_\ell^{(1)}(k\,r_>) \stackrel{k\to 0}{=} \frac{1}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}}\,. \tag{3.91}
$$

Using the asymptotics

$$
j_\ell(z) \sim \frac{z^\ell}{(2\ell+1)!!}\,, \qquad y_\ell(z) \sim -\frac{(2\ell-1)!!}{z^{\ell+1}}\,, \tag{3.92}
$$

we can establish that

$$
\begin{aligned}
\mathrm{i}\,k\,j_\ell(k\,r_<)\,h_\ell^{(1)}(k\,r_>) &= \mathrm{i}\,k\,j_\ell(k\,r_<)\,\left(j_\ell(k\,r_>) + \mathrm{i}y_\ell(k\,r_>)\right) \\
&\stackrel{k\to 0}{=} -\,k\,j_\ell(k\,r_<)\,y_\ell(k\,r_>) \\
&\stackrel{k\to 0}{=} -\,k\,\frac{(k\,r_<)^\ell}{(2\ell+1)!!}\left(-\frac{(2\ell-1)!!}{(kr_>)^{\ell+1}}\right) = \frac{1}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}}\,.
\end{aligned} \tag{3.93}
$$

The general forms for $\vec{\Phi}_0(\vec{r})$ and $\vec{A}_0(\vec{r})$ are

$$
\Phi_0(\vec{r}) = \int \mathrm{d}^3r'\,\rho_0(\vec{r}\,')\,\frac{\exp\left(\mathrm{i}\omega|\vec{r}-\vec{r}\,'|/c\right)}{4\pi\epsilon_0c^2\,|\vec{r}-\vec{r}\,'|}\,, \tag{3.94a}
$$

$$
\vec{A}_0(\vec{r}) = \int \mathrm{d}^3r'\,\vec{J}_0(\vec{r}\,')\,\frac{\exp\left(\mathrm{i}\omega|\vec{r}-\vec{r}\,'|/c\right)}{4\pi\epsilon_0c^2\,|\vec{r}-\vec{r}\,'|}\,, \tag{3.94b}
$$

and thus

$$\vec{A}_0(\vec{r}) = \frac{\mathrm{i}k}{\epsilon_0 c^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi) \int \vec{J}_0(\vec{r}')\, j_\ell(k\,r_<)\, h_\ell^{(1)}(k\,r_>)\, Y_{lm}^*(\theta',\varphi')\, r'^2\, \mathrm{d}r' \mathrm{d}\Omega' \,. \tag{3.95}$$

In all following considerations, we shall assume that the source current is localized near $r' = 0$, so that $r_< = r'$, and $r_> = r$,

Vector potential for an extended source:

$$\vec{A}_0(\vec{r}) = \frac{\mathrm{i}k}{\epsilon_0 c^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} h_\ell^{(1)}(k\,r)\, Y_{\ell m}(\theta,\varphi) \int \vec{J}_0(\vec{r}')\, j_\ell(k\,r')\, Y_{\ell m}^*(\theta',\varphi')\, r'^2\, \mathrm{d}r' \mathrm{d}\Omega' \,. \tag{3.96}$$

Approximating $j_\ell(k\,r')$ with the first term in the asymptotic form near $r' = 0$,

$$j_\ell(z) \sim \frac{z^\ell}{(2\ell+1)!!}\,, \qquad z \to 0\,, \tag{3.97}$$

one obtains

$$\vec{A}_0(\vec{r}) \approx \frac{\mathrm{i}k}{\epsilon_0 c^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} h_\ell^{(1)}(k\,r)\, Y_{\ell m}(\theta,\varphi) \int \vec{J}_0(\vec{r}')\, \frac{1}{(2\ell+1)!!}\, k^\ell\, r'^{\ell+2}\, Y_{\ell m}^*(\theta',\varphi')\, \mathrm{d}r'\, \mathrm{d}\Omega' \tag{3.98}$$

and so

Vector potential for a localized source:

$$\vec{A}_0(\vec{r}) \approx \frac{\mathrm{i}k}{\epsilon_0 c^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{k^\ell\, h_\ell^{(1)}(k\,r)}{(2\ell+1)!!}\, Y_{\ell m}(\theta,\varphi) \int \vec{J}_0(\vec{r}')\, r'^{\ell+2}\, Y_{\ell m}^*(\theta',\varphi')\, \mathrm{d}r'\, \mathrm{d}\Omega' \,. \tag{3.99}$$

In the "near field" region, closer than a wavelength away from the antenna, we have

$$d < r \ll \frac{c}{\omega}\,, \qquad k\,r \ll 1\,, \qquad h_\ell^{(1)}(k\,r) \approx +\mathrm{i}\, y_\ell(k\,r) \approx -\mathrm{i}\, \frac{(2\ell-1)!!}{(k\,r)^{\ell+1}}\,. \tag{3.100}$$

So,

$$\vec{A}_0(\vec{r}) = \frac{k}{\epsilon_0 c^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{(2\ell-1)!!}{(kr)^{\ell+1}}\, \frac{k^\ell}{(2\ell+1)!!}\, Y_{\ell m}(\theta,\varphi) \int \vec{J}_0(\vec{r}')\, r'^{\ell+2}\, Y_{\ell m}^*(\theta',\varphi')^*\, \mathrm{d}r'\, \mathrm{d}\Omega' \tag{3.101}$$

or

Vector potential for a localized source/near field zone:

$$\vec{A}_0(\vec{r}) = \frac{1}{\epsilon_0 c^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{Y_{\ell m}(\theta,\varphi)}{(2\ell+1)!!\, r^{\ell+1}} \int \vec{J}_0(\vec{r}')\, r'^{\ell+2}\, Y_{\ell m}(\theta',\varphi')^*\, \mathrm{d}r' \mathrm{d}\Omega' \,. \tag{3.102}$$

In the far zone,

$$r \gg \frac{c}{\omega} = \frac{1}{k}\,, \qquad h_\ell^{(1)}(k\,r) \approx -\mathrm{i}\, \frac{\exp\left[\mathrm{i}\left(k\,r - \frac{\ell\pi}{2}\right)\right]}{k\,r}\,. \tag{3.103}$$

The radiation form for $\vec{A}_0(\vec{r})$ is approximated using

Vector potential for a localized source/radiation zone:

$$\vec{A}_0(\vec{r}) \approx \frac{k}{\epsilon_0\, c^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{k^\ell}{(2\ell+1)!!} \frac{\exp\left[\mathrm{i}\left(k\,r - \frac{\pi\ell}{2}\right)\right]}{kr}\, Y_{\ell m}(\theta,\varphi) \int \vec{J}_0(\vec{r}\,')\; r'^{\ell+2}\, Y^*_{\ell m}(\theta',\varphi')\,\mathrm{d}r'\mathrm{d}\Omega'$$
$$= \frac{1}{\epsilon_0\, c^2\, r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{k^\ell}{(2\ell+1)!!} \exp\left[\mathrm{i}\left(k\,r - \frac{\pi\ell}{2}\right)\right]\; Y_{\ell m}(\theta,\varphi) \int \vec{J}_0(\vec{r}\,')\; r'^{\ell+2}\, Y^*_{\ell m}(\theta',\varphi')\,\mathrm{d}r'\mathrm{d}\Omega'\,. \tag{3.104}$$

By dimensional analysis, the source integral is proportional to

$$\int \vec{J}_0(\vec{r}\,')\; r'^{\ell+2}\, Y^*_{\ell m}(\theta',\varphi')\,\mathrm{d}r'\mathrm{d}\Omega' \sim j_0\, d^{\ell+3}\,, \tag{3.105}$$

where j_0 is a characteristic scale of the current density. The expansion in multipoles thus is seen to be an expansion in the parameter

$$k\,d \ll 1 \tag{3.106}$$

and can be evaluated term by term. The lowest nonvanishing term then gives the dominant contribution. The multipole formalism also is useful if $\vec{J}_0(\vec{r})$ is described by a superposition of one or two spherical harmonics.

If many spherical harmonics contribute, one might otherwise approximate, with $\hat{r} = \vec{r}/r$,

$$|\vec{r}-\vec{r}\,'| = r\sqrt{\left(\hat{r}-\frac{\vec{r}\,'}{r}\right)^2} = r\left(1 - \frac{1}{2}\,2\,\frac{\hat{r}\cdot\vec{r}\,'}{r} + \ldots\right) = r - \frac{\vec{r}\cdot\vec{r}\,'}{r} + \ldots\,, \qquad r\to\infty\,, \tag{3.107}$$

and thus

$$\vec{A}(\vec{r},t) \approx \frac{\exp(\mathrm{i}\,k\,r - \mathrm{i}\,\omega\,t)}{4\pi\epsilon_0 c^2} \int \mathrm{d}^3r'\,\vec{J}_0(\vec{r}\,')\;\frac{1}{|\vec{r}-\vec{r}\,'|}\,\exp\left(-\mathrm{i}\,k\,\frac{\vec{r}\,'\cdot\vec{r}}{r}\right)\,\mathrm{d}^3r'$$
$$\approx \frac{\exp(\mathrm{i}\,k\,r - \mathrm{i}\,\omega\,t)}{4\pi\epsilon_0 c^2}\,\frac{1}{r}\int \vec{J}_0(\vec{r}\,')\;\exp\left(-\mathrm{i}\,k\,\frac{\vec{r}\,'\cdot\vec{r}}{r}\right)\,\mathrm{d}^3r'\,. \tag{3.108}$$

The latter expression contains, in some sense, a sum over all multipoles, but is valid only in the limit $r/r' \to \infty$, whereas Eq. (3.104) only makes the assumption that r needs to lie outside the area where the current distribution $J_0(\vec{r})$ is nonvanishing.

3.3.2 Dipole Radiation

We first consider the dipole term, i.e., the $\ell = 0$ term in the expansion. In the expansion of the vector potential, this is the term with $\ell = 0$ and $m = 0$. The integral over the current density is converted to one over the charge density using the identity

$$\vec{\nabla}\cdot\left(x_m \vec{J}_0(\vec{r})\right) = \sum_{n=1}^{3}\frac{\partial}{\partial x_n}x_m\,J_{0,n}(\vec{r}) = \sum_{n=1}^{3} J_{0,n}(\vec{r})\;\delta_{n\,m} + x_m\vec{\nabla}\cdot\vec{J}_0(\vec{r})$$
$$= J_{0,m}(\vec{r}) + \mathrm{i}\,\omega\,x_m\,\rho_0(\vec{r})\,, \tag{3.109}$$

because the charge conservation reads as $\vec{\nabla}\cdot\vec{J}_0(\vec{r}) = -\left(-\mathrm{i}\,\omega\rho_0(\vec{r})\right)$ in Fourier space.

The general paradigm for the multipole decomposition is this: In the first step, one writes an expression with, say, for the quadrupole component of the charge density, *two* coordinates in front of the current density vector. Then, one forms the total differential (the integral of it vanishes by virtue of the Gauss law). Acting with the differential gives two kinds of expressions, the first of which contains one coordinate less, and the second of which contains the divergence of the current density, i.e., the charge density, with (again, in the quadrupole case) two coordinates. One then has an equivalent formulation, for expressions with either two coordinates and the charge density (which is the desired formula) or one coordinate (in general, one coordinate less than the first term) and the current density. The latter expression is thus reformulated in terms of the first.

The integral over the $\vec{J}_0(\vec{r}\,')$ for $\ell = 0$ projects out only the spherically symmetric part of the $\vec{J}_0(r', \theta', \varphi')$,

$$\int \vec{J}_0(\vec{r}\,')\; r'^2\, Y_{00}(\theta', \varphi')^*\, \mathrm{d}r' \mathrm{d}\Omega' = \frac{1}{\sqrt{4\pi}} \int \vec{J}_0(\vec{r}\,')\; \mathrm{d}^3 r' \,. \tag{3.110}$$

Because the integral of $\vec{\nabla} \cdot \left(x_m \vec{J}_0(\vec{r}\,') \right)$ over all space vanishes in view of the divergence theorem, we have

$$\begin{aligned} \frac{1}{\sqrt{4\pi}} \int \vec{J}_0(\vec{r}\,')\; \mathrm{d}^3 r' &= -\,\mathrm{i}\frac{\omega}{\sqrt{4\pi}} \int \sum_{m=1}^{3} \hat{e}_m\, x'_m\, \rho_0(\vec{r}\,')\, \mathrm{d}^3 r' = -\mathrm{i}\frac{\omega}{\sqrt{4\pi}} \int \vec{r}\,'\; \rho_0(\vec{r}\,')\, \mathrm{d}^3 r' \\ &= -\,\mathrm{i}\frac{\omega}{\sqrt{4\pi}} \int \vec{r}\,' \rho_0(\vec{r}\,')\, \mathrm{d}^3 r' = -\mathrm{i}\frac{\omega}{\sqrt{4\pi}}\, \vec{p}_0 \,. \end{aligned} \tag{3.111}$$

This last integral is the electric dipole moment of the charge distribution, i.e., $\vec{p}_0$. We recall Eq. (3.104), which for the current case holds for all distances from the source, in view of Eq. (3.37) and (3.38),

$$\vec{A}_0(\vec{r}) \approx \frac{1}{\epsilon_0\, c^2\, r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{k^\ell}{(2\ell+1)!!} \exp\left[\mathrm{i}\left(k\,r - \frac{\pi \ell}{2}\right)\right]\; Y_{\ell m}(\theta, \varphi) \int \vec{J}_0(\vec{r}\,')\; r'^{\ell+2}\; Y^*_{\ell m}(\theta', \varphi')\, \mathrm{d}r' \mathrm{d}\Omega' \,. \tag{3.112}$$

(We are still working in the Lorenz gauge, here.) The $\ell = 0$ component amounts to

$$\begin{aligned} \vec{A}_0(\vec{r}) &\sim \frac{k}{\epsilon_0\, c^2} \frac{\exp(\mathrm{i}k\,r)}{kr} \frac{1}{\sqrt{4\pi}} \left(\int \vec{J}_0(\vec{r}\,')\; r'^2\; \frac{1}{\sqrt{4\pi}} \mathrm{d}r' \mathrm{d}\Omega' \right) \\ &= \frac{k}{4\pi\epsilon_0\, c^2} \frac{\exp(\mathrm{i}k\,r)}{kr}\; (-\mathrm{i}\omega\, \vec{p}_0) \,, \end{aligned} \tag{3.113}$$

and the vector potential for the radiating dipole thus is

$$\boxed{\textbf{Vector Potential for Dipole:}} \qquad \vec{A}_0(\vec{r}) = -\mathrm{i}\, \frac{k\, \vec{p}_0}{4\pi\epsilon_0 c} \frac{\exp(\mathrm{i}kr)}{r} \,. \tag{3.114}$$

The magnetic induction generated by an oscillating electric dipole is

$$\begin{aligned} \vec{B}_0(\vec{r}) &= \vec{\nabla} \times \vec{A}_0(\vec{r}) = \frac{k}{4\pi\epsilon_0\, c} \left(-\mathrm{i}\, \vec{\nabla} \times \vec{p}_0 \right) \left(\frac{\exp(\mathrm{i}\,k\,r)}{r} \right) = \frac{k}{4\pi\epsilon_0\, c} \left(\mathrm{i}\, \vec{p}_0 \times \vec{\nabla} \right) \left(\frac{\exp(\mathrm{i}\,k\,r)}{r} \right) \\ &= \mathrm{i}\frac{k}{4\pi\epsilon_0\, c} \left(\vec{p}_0 \times \vec{\nabla} r \right) \left[\frac{\mathrm{d}}{\mathrm{d}r} \frac{\exp(\mathrm{i}\,k\,r)}{r} \right] = -\mathrm{i}\frac{k}{4\pi\epsilon_0\, c} \left(\frac{\vec{r}}{r} \times \vec{p}_0 \right) \left[\frac{\exp(\mathrm{i}k\,r)}{r} \right] \left(\mathrm{i}k - \frac{1}{r} \right) \\ &= -\,\mathrm{i}\frac{k^2}{4\pi\epsilon_0\, c} \left(\frac{\vec{r}}{r} \times \vec{p}_0 \right) \left[\frac{\exp(\mathrm{i}k\,r)}{r} \right] \left(\mathrm{i} - \frac{1}{kr} \right) \\ &= \frac{k^2}{4\pi\epsilon_0\, c} \left(\frac{\vec{r}}{r} \times \vec{p}_0 \right) \left[\frac{\exp(\mathrm{i}k\,r)}{r} \right] \left(1 - \frac{1}{\mathrm{i}\,k\,r} \right) . \end{aligned} \tag{3.115}$$

The direction of the magnetic is perpendicular to the axis defined by the dipole. The leading term, for very large r, is

Mag. Field, Dipole, Radiation Zone:

$$\vec{B}_0\,(\vec{r}) \approx \frac{k^2}{4\pi\epsilon_0\, c}\ (\hat{r} \times \vec{p}_0) \left[\frac{\exp\,(\mathrm{i}k\,r)}{r}\right] \qquad \text{for} \qquad k\,r \gg 1\,. \tag{3.116}$$

In order to find the electric field in the radiation zone, $k\,r \gg 1$, we recall the Maxwell equations,

$$\vec{\nabla}\cdot\vec{E}_0(\vec{r}) = \frac{1}{\epsilon_0}\,\rho_0(\vec{r})\,, \qquad \vec{\nabla}\cdot\vec{B}_0(\vec{r}) = 0\,,$$
$$\vec{\nabla}\times\vec{E}_0(\vec{r}) - \mathrm{i}\omega\vec{B}_0(\vec{r}) = 0\,, \qquad \vec{\nabla}\times\vec{B}_0(\vec{r}) + \frac{\mathrm{i}\omega}{c^2}\vec{E}_0(\vec{r}) = \mu_0\,\vec{J}_0(\vec{r})\,. \tag{3.117}$$

Thus,

$$\begin{aligned}\vec{E}_0\,(\vec{r}) &= \frac{\mathrm{i}c^2}{\omega}\ \left(\vec{\nabla}\times\vec{B}_0\,(\vec{r})\right) = \mathrm{i}\,\frac{c}{k}\ \left(\vec{\nabla}\times\vec{B}_0\,(\vec{r})\right)\\ &= \frac{\mathrm{i}k}{4\pi\epsilon_0}\vec{\nabla}\times\left[\left(\frac{\vec{r}}{r}\times\vec{p}_0\right)f(r)\right] \qquad \text{where} \qquad f(r) = \frac{\exp\,(\mathrm{i}k\,r)}{r}\left(1 - \frac{1}{\mathrm{i}k\,r}\right)\,.\end{aligned} \tag{3.118}$$

The leading term, for large r, is generated by the gradient operator pulling down a factor $\vec{k}$ from the exponential, and can be written as

$$\begin{aligned}\vec{E}_0\,(\vec{r}) &\approx \frac{\mathrm{i}k}{4\pi\epsilon_0}\left[\left(\vec{\nabla}\,r\right)\times\left(\frac{\vec{r}}{r}\times\vec{p}_0\right)\right]\ \frac{\mathrm{d}}{\mathrm{d}r}f(r) \qquad\qquad (k\,r\gg 1)\\ &\approx \frac{\mathrm{i}k}{4\pi\epsilon_0}\ \left[\frac{\vec{r}}{r}\times\left(\frac{\vec{r}}{r}\times\vec{p}_0\right)\right]\ \frac{\mathrm{i}k\ \exp\,(\mathrm{i}k\,r)}{r}\\ &\approx\ -\frac{k^2}{4\pi\epsilon_0}\ \left[\frac{\vec{r}}{r}\times\left(\frac{\vec{r}}{r}\times\vec{p}_0\right)\right]\ \frac{\exp\,(\mathrm{i}k\,r)}{r}\\ &\approx \frac{k^2}{4\pi\epsilon_0}\ \left[(\hat{r}\times\vec{p}_0)\times\hat{r}\right]\ \frac{\exp\,(\mathrm{i}k\,r)}{r}\,,\end{aligned} \tag{3.119}$$

and so

Elec. Field, Dipole, Radiation Zone:

$$\vec{E}_0\,(\vec{r}) \approx \frac{k^2}{4\pi\epsilon_0}\ \left[(\hat{r}\times\vec{p}_0)\times\hat{r}\right]\ \frac{\exp\,(\mathrm{i}k\,r)}{r} \qquad \text{for} \qquad k\,r \gg 1\,. \tag{3.120}$$

This field, in contrast to the magnetic field, is in the direction of the dipole. Together with Eq. (3.116),

$$\vec{B}_0\,(\vec{r}) \approx \frac{k^2}{4\pi\epsilon_0\,c}\ (\hat{r}\times\vec{p}_0)\ \frac{\exp\,(\mathrm{i}k\,r)}{r} \qquad \text{for} \qquad k\,r \gg 1\,. \tag{3.121}$$

one infers that

$$\vec{E}_0\,(\vec{r}) \approx c\,\vec{B}_0\,(\vec{r})\times\hat{r} \tag{3.122}$$

in the radiation zone (the electric field, the magnetic field as well as the radius vector $\vec{r}\parallel\vec{E}_0\times\vec{B}_0$ form a right-handed system). The important identity

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}\,(\vec{a}\cdot\vec{c}) - \vec{c}\,(\vec{a}\cdot\vec{b}) \tag{3.123}$$

can be used to calculate the Poynting vector

$$
\begin{aligned}
\left\langle \vec{S}(\vec{r}) \right\rangle &= \frac{1}{2\mu_0} \vec{E}_0(\vec{r}) \times B_0^*(\vec{r}) = \frac{1}{2\mu_0} \frac{k^2}{4\pi\epsilon_0} \, \frac{k^2}{4\pi\epsilon_0 \, c} \, \frac{1}{r^2} \, \left[(\hat{r} \times \vec{p}_0) \times \hat{r} \right] \times (\hat{r} \times \vec{p}_0) \\
&= -\frac{c}{2\mu_0 c^2} \frac{k^4}{(4\pi\epsilon_0)^2} \, \frac{1}{r^2} \, (\hat{r} \times \vec{p}_0) \times \left[(\hat{r} \times \vec{p}_0) \times \hat{r} \right] = \frac{\epsilon_0 c}{2} \frac{k^4}{(4\pi\epsilon_0)^2} \, \frac{1}{r^2} \, \hat{r} \, \left(\hat{r} \times \vec{p}_0 \right)^2 \\
&= \frac{\epsilon_0 c}{2} \frac{k^4}{(4\pi\epsilon_0)^2} \, \frac{1}{r^2} \, \hat{r} \, \left(\vec{p}_0^{\,2} - (\hat{r} \cdot \vec{p}_0)^2 \right) = \frac{\epsilon_0 c}{2} \frac{k^4 \vec{p}_0^{\,2}}{(4\pi\epsilon_0)^2} \, \frac{1}{r^2} \, \hat{r} \, \left(1 - \cos^2\theta \right) , \qquad (3.124)
\end{aligned}
$$

where $\vec{p}_0^{\,2} = \vec{p}_0 \cdot \vec{p}_0^{\,*}$ in the case of a complex dipole vector. The angular distribution of the average radiated power $\mathrm{d}P_{\mathrm{avg}}$ per area $\mathrm{d}A$ is

$$
\frac{\mathrm{d}P_{\mathrm{avg}}\,(\Omega)}{\mathrm{d}A} = \hat{r} \cdot \overline{S}(\vec{r}) = \frac{\epsilon_0 c}{2} \frac{k^4 \, \vec{p}_0^{\,2}}{(4\pi\epsilon_0)^2} \, \frac{1}{r^2} \, \left(1 - \cos^2\theta \right) . \qquad (3.125)
$$

The average intensity radiated parallel to the dipole vector is seen to be zero. The total, time averaged, power radiated by an oscillating electric dipole is obtained with $\mathrm{d}A = r^2 \, \mathrm{d}\Omega$ as

$$
P_{\mathrm{avg}} = \frac{\epsilon_0 c}{2} \frac{k^4 \, \vec{p}_0^{\,2}}{(4\pi\epsilon_0)^2} \int \frac{1}{r^2} \, \left(1 - \cos^2\theta \right) r^2 \, \mathrm{d}\Omega = \frac{c}{3} \, \frac{k^4 \, \vec{p}_0^{\,2}}{4\pi\epsilon_0} \, . \qquad (3.126)
$$

This average power penetrates the cross-sectional area given by the surface of the sphere of radius r. The radiated power is proportional to the fourth power of the frequency of the oscillation.

3.3.3 Exact Expression for the Radiating Dipole

We had seen that

$$
\vec{B}_0(\vec{r}) = \frac{k^2}{4\pi\epsilon_0 \, c} \, \left(\frac{\vec{r}}{r} \times \vec{p}_0 \right) \left[\frac{\exp{(\mathrm{i}k \, r)}}{r} \right] \, \left(1 - \frac{1}{\mathrm{i} \, k \, r} \right) \qquad (3.127)
$$

and the electric field was obtained as

$$
\vec{E}_0\,(\vec{r}) = \frac{\mathrm{i}c^2}{\omega} \, \left(\vec{\nabla} \times \vec{B}_0\,(\vec{r}) \right) . \qquad (3.128)
$$

We had previously found in Eq. (3.120) that

$$
\vec{E}_0\,(\vec{r}) \approx \frac{k^2}{4\pi\epsilon_0} \, \left[(\hat{r} \times \vec{p}_0) \times \hat{r} \right] \, \frac{\exp{(\mathrm{i}k \, r)}}{r} \qquad \text{for} \qquad k \, r \gg 1 \, . \qquad (3.129)
$$

The exact expression for $\vec{E}_0(\vec{r})$ remains to be derived. Certainly,

$$
\begin{aligned}
\vec{E}_0\,(\vec{r}) &= \frac{\mathrm{i}c}{k} \, \frac{k^2}{4\pi\epsilon_0 \, c} \, \left(\vec{\nabla} \times \left(\frac{\vec{r}}{r} \times \vec{p}_0 \right) \frac{\exp{(\mathrm{i}k \, r)}}{r} \, \left(1 - \frac{1}{\mathrm{i} \, k \, r} \right) \right) \\
&= \frac{\mathrm{i}k}{4\pi\epsilon_0} \, \left(\vec{\nabla} \times \left[(\vec{r} \times \vec{p}_0) \, \frac{\exp{(\mathrm{i}k \, r)}}{r^2} \, \left(1 - \frac{1}{\mathrm{i} \, k \, r} \right) \right] \right) \\
&= \frac{\mathrm{i}k}{4\pi\epsilon_0} \, \left[(\vec{\nabla} r) \times \left(\frac{\vec{r}}{r} \times \vec{p}_0 \right) \right] \, r \, \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{\exp{(\mathrm{i}k \, r)}}{r^2} \, \left(1 - \frac{1}{\mathrm{i} \, k \, r} \right) \right) \\
&\quad + \frac{\mathrm{i}k}{4\pi\epsilon_0} \, \left(\frac{\exp{(\mathrm{i}k \, r)}}{r^2} \, \left(1 - \frac{1}{\mathrm{i} \, k \, r} \right) \right) \, \left(\vec{\nabla} \times (\vec{r} \times \vec{p}_0) \right) . \qquad (3.130)
\end{aligned}
$$

We have replaced $\vec{r} \to \vec{r}/r$ in the cross product and multiplied through by r, and also, we have used the general formula

$$\vec{\nabla} \times \left(\vec{A}(\vec{r})\, B(\vec{r})\right) = \left[\vec{\nabla} \times \vec{A}(\vec{r})\right] \, B(\vec{r}) - \vec{A}(\vec{r}) \times \vec{\nabla} B(\vec{r}) \,, \tag{3.131}$$

which is valid for general vector-valued function $\vec{A} = \vec{A}(\vec{r})$ and scalar functions $B = B(\vec{r})$. Furthermore, according to the rule that $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\,(\vec{a} \cdot \vec{b})$,

$$\left(\vec{\nabla} \times (\vec{r} \times \vec{p}_0)\right) = \left(\vec{\nabla} \cdot \vec{p}_0\right) \vec{r} - \vec{p}_0 \left(\vec{\nabla} \cdot \vec{r}\right) = \vec{p}_0 - 3\,\vec{p}_0 = -2\vec{p}_0 \,. \tag{3.132}$$

The rule (3.123) would otherwise suggest that the first term should read $\vec{r}\left(\vec{\nabla} \cdot \vec{p}_0\right)$, suggesting that it vanishes, but in this particular case, one would have to consider a gradient operator $\vec{\nabla}$ acting to the left, in the sense of the replacement $\vec{\nabla} \to \overleftarrow{\vec{\nabla}}$.

We have taken into account the fact that the $\vec{\nabla}$ operator acts on everything to the right. So,

$$\begin{aligned}
\vec{E}_0(\vec{r}) &= \frac{\mathrm{i}k}{4\pi\epsilon_0} \left[\frac{\vec{r}}{r} \times \left(\frac{\vec{r}}{r} \times \vec{p}_0\right)\right] \left[r \, \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{\exp(\mathrm{i}k\, r)}{r^2} \left(1 - \frac{1}{\mathrm{i}\, k\, r}\right)\right)\right] \\
&\quad + \frac{\mathrm{i}k}{4\pi\epsilon_0} \left(\frac{\exp(\mathrm{i}k\, r)}{r^2} \left(1 - \frac{1}{\mathrm{i}\, k\, r}\right)\right) (-2\vec{p}_0) \\
&= \frac{\mathrm{i}k}{4\pi\epsilon_0} \left[\hat{r} \times (\hat{r} \times \vec{p}_0)\right] \left(\frac{\mathrm{i}k}{r} - \frac{3}{r^2} - \frac{3\mathrm{i}}{kr^3}\right) \exp(\mathrm{i}k\, r) + \frac{\mathrm{i}k}{4\pi\epsilon_0} \left(\frac{\exp(\mathrm{i}k\, r)}{r^2} \left(1 - \frac{1}{\mathrm{i}\, k\, r}\right)\right) (-2\vec{p}_0) \\
&= \frac{\mathrm{i}k}{4\pi\epsilon_0} \left[\hat{r} \times (\hat{r} \times \vec{p}_0)\right] \frac{\mathrm{i}k}{r} \exp(\mathrm{i}k\, r) + \frac{\mathrm{i}k}{4\pi\epsilon_0} \left[\hat{r} \times (\hat{r} \times \vec{p}_0)\right] \left(-\frac{3}{r^2} - \frac{3\mathrm{i}}{kr^3}\right) \exp(\mathrm{i}k\, r) + \\
&\quad + \frac{\mathrm{i}k}{4\pi\epsilon_0} \left(\frac{\exp(\mathrm{i}k\, r)}{r^2} \left(1 - \frac{1}{\mathrm{i}\, k\, r}\right)\right) (-2\vec{p}_0) \,.
\end{aligned} \tag{3.133}$$

Finally,

$$\begin{aligned}
\vec{E}_0(\vec{r}) &= \frac{k^2}{4\pi\epsilon_0} \left[(\hat{r} \times \vec{p}_0) \times \hat{r}\right] \frac{\exp(\mathrm{i}k\, r)}{r} + \frac{1}{4\pi\epsilon_0} \left[\hat{r}\,(\hat{r} \cdot \vec{p}_0 - \vec{p}_0)\right] \left(-\frac{3\mathrm{i}k}{r^2} + \frac{3}{r^3}\right) \exp(\mathrm{i}k\, r) + \\
&\quad + \frac{1}{4\pi\epsilon_0} \left(\frac{\exp(\mathrm{i}k\, r)}{r^2} \left(\mathrm{i}k - \frac{1}{r}\right)\right) (-2\vec{p}_0) \,.
\end{aligned} \tag{3.134}$$

The first term is the leading term. The remaining terms have the structure of a quadrupole term component when projected onto the dipole vector $\vec{p}_0$,

$$\begin{aligned}
\vec{E}_0(\vec{r}) &= \frac{k^2}{4\pi\epsilon_0} \left[(\hat{r} \times \vec{p}_0) \times \hat{r}\right] \frac{\exp(\mathrm{i}k\, r)}{r} + \frac{1}{4\pi\epsilon_0} \left[3\hat{r}\,(\hat{r} \cdot \vec{p}_0 - 3\vec{p}_0)\right] \left(\frac{1}{r^3} - \frac{\mathrm{i}k}{r^2}\right) \exp(\mathrm{i}k\, r) \\
&\quad + \frac{1}{4\pi\epsilon_0} \exp(\mathrm{i}k\, r) \left(-\frac{\mathrm{i}k}{r^2} + \frac{1}{r^3}\right) (+2\vec{p}_0) \\
&= \frac{k^2}{4\pi\epsilon_0} \left[(\hat{r} \times \vec{p}_0) \times \hat{r}\right] \frac{\exp(\mathrm{i}k\, r)}{r} + \frac{1}{4\pi\epsilon_0} \left[3\hat{r}\,(\hat{r} \cdot \vec{p}_0 - 3\vec{p}_0)\right] \left(\frac{1}{r^3} - \frac{\mathrm{i}k}{r^2}\right) \exp(\mathrm{i}k\, r) \\
&\quad + \frac{1}{4\pi\epsilon_0} (2\vec{p}_0) \left(\frac{1}{r^3} - \frac{\mathrm{i}k}{r^2}\right) \exp(\mathrm{i}k\, r) \\
&= \frac{k^2}{4\pi\epsilon_0} \left[(\hat{r} \times \vec{p}_0) \times \hat{r}\right] \frac{\exp(\mathrm{i}k\, r)}{r} + \frac{1}{4\pi\epsilon_0} \left[3\hat{r}\,(\hat{r} \cdot \vec{p}_0) - \vec{p}_0\right] \left(\frac{1}{r^3} - \frac{\mathrm{i}k}{r^2}\right) \exp(\mathrm{i}k\, r) \,.
\end{aligned} \tag{3.135}$$

This is formula (9.18) in Chapter 9 of [J. D. Jackson, *Classical Electrodynamics*, (John Wiley and Sons, New York, 1998)]. Let us compile our exact results,

Mag. Field, Dipole, Exact: $$\vec{B}_0(\vec{r}) = \frac{\mathrm{i}k}{4\pi\epsilon_0\, c} \, (\hat{r} \times \vec{p}_0) \left(\frac{1}{r^2} - \frac{\mathrm{i}k}{r}\right) \exp(\mathrm{i}k\, r) \,, \tag{3.136}$$

and

$$\boxed{\textbf{Elec. Field, Dipole, Exact:}}$$

$$\vec{E}_0(\vec{r}) = \frac{k^2}{4\pi\epsilon_0} \, [(\hat{r} \times \vec{p}_0) \times \hat{r}] \, \frac{\exp(\mathrm{i}k\,r)}{r} + \frac{1}{4\pi\epsilon_0} \, [3\hat{r}\,(\hat{r} \cdot \vec{p}_0) - \vec{p}_0] \left(\frac{1}{r^3} - \frac{\mathrm{i}k}{r^2} \right) \exp(\mathrm{i}k\,r) \, . \tag{3.137}$$

In the near field, $kr \ll 1$, we can replace $\exp(\mathrm{i}k\,r) \to 1$ and find

$$\boxed{\textbf{Mag. Field, Dipole, Near Zone:}} \qquad \vec{B}_0(\vec{r}) \approx \frac{\mathrm{i}k}{4\pi\epsilon_0\,c} \, (\hat{r} \times \vec{p}_0) \, \frac{1}{r^2} \, , \qquad k\,r \ll 1 \tag{3.138}$$

and

$$\boxed{\textbf{Elec. Field, Dipole, Near Zone:}} \qquad \vec{E}_0(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \, [3\hat{r}\,(\hat{r} \cdot \vec{p}_0 - \vec{p}_0)] \, \frac{1}{r^3} \, , \qquad k\,r \ll 1 \, . \tag{3.139}$$

The interested reader may want to verify all steps leading to the exact results for the dipole radiation, with all intermediate steps clearly written out, including the differentiations not done explicitly here.

3.4 Tensor Green Function

3.4.1 Clebsch–Gordan Coefficients: Motivation

Angular momentum algebra is connected to the so-called Clebsch–Gordan or vector addition coefficients. These coefficients are of rather universal applicability over wide ranges of physical theory. One uses them in order to find the expansion coefficients of a tensor of higher rank as it is composed out of elements of tensors of lower rank, hence the name “vector” addition coefficients (while “tensor” addition might otherwise be a more precise formulation).

One of the most important symmetries in physics concerns the group of rotations, or, the special orthogonal group in three dimensions, called $\mathrm{SO}(3)$. We know that the scalar product $\vec{u} \cdot \vec{v}$ of two vectors is invariant under rotations. The vector product $\vec{u} \times \vec{v}$ transforms as a (pseudo-)vector iteself. We recall that a pseudo-vector, in contrast to a vector, conserves its sign under parity, $\vec{u} \times \vec{v} \to (-\vec{u}) \times (-\vec{v})$. Vectors do not mix with scalars under rotations. The vectors are of rank one, whereas scalars are tensors of rank zero. Furthermore, from the tensor product of two vectors, we can extract a third quantity which is a quadrupole tensor of rank two, which also does not mix with tensors of different rank under rotations. From the tensor product of two vectors, one may thus extract tensors of rank zero, one, and two. The components of these “constructed” tensors are linear combinations of products of the components of the vectors. The corresponding formalism, in the spherical basis, involves the vector addition, or Clebsch–Gordan, coefficients.

The procedure is illustrated most effectively by way of an example. We consider a second-rank tensor from the tensor product of two vectors,

$$\mathbb{T} = \vec{u} \otimes \vec{v} = \begin{pmatrix} u_x\,v_x & u_x\,v_y & u_x\,v_z \\ u_y\,v_x & u_y\,v_y & u_y\,v_z \\ u_z\,v_x & u_z\,v_y & u_z\,v_z \end{pmatrix} . \tag{3.140}$$

We can decompose $\mathbb{T}$ as follows,

$$\mathbb{T} = \left.\mathbb{T}\right|_{\ell=0} + \left.\mathbb{T}\right|_{\ell=1} + \left.\mathbb{T}\right|_{\ell=2} \,, \tag{3.141}$$

$$\left.\mathbb{T}\right|_{\ell=0} = \frac{\mathrm{trc}(\mathbb{T})}{3}\, \mathbb{1}_{3\times 3} \,, \qquad \mathrm{trc}(\mathbb{T}) = u_x\, v_x + u_y\, v_y + u_z\, v_z \,, \tag{3.142}$$

$$\left.\mathbb{T}\right|_{\ell=1} = \frac{1}{2}\left(\mathbb{T} - \mathbb{T}^{\mathrm{T}}\right) = \begin{pmatrix} 0 & \frac{1}{2}\,(u_x\, v_y - u_y\, v_x) & -\frac{1}{2}\,(u_z\, v_x - u_x\, v_z) \\ -\frac{1}{2}\,(u_x\, v_y - u_y\, v_x) & 0 & \frac{1}{2}\,(u_y\, v_z - u_z\, v_y) \\ \frac{1}{2}\,(u_z\, v_x - u_x\, v_z) & -\frac{1}{2}\,(u_y\, v_z - u_z\, v_y) & 0 \end{pmatrix} . \tag{3.143}$$

The $(\ell = 0)$-component is invariant under rotations. The entries of the matrix $\left.\mathbb{T}\right|_{\ell=1}$ can be identified as the components of the vector product of $\vec{u}$ and $\vec{v}$, with details to be discussed below. Finally, the $(\ell = 2)$-component reads as

$$\left.\mathbb{T}\right|_{\ell=2} = \frac{1}{2}\left(\mathbb{T} + \mathbb{T}^{\mathrm{T}}\right) - \frac{\mathrm{trc}(\mathbb{T})}{3}\, \mathbb{1}_{3\times 3} \tag{3.144}$$

$$= \begin{pmatrix} \dfrac{2u_x\, v_x - u_y\, v_y - u_z\, v_z}{3} & \dfrac{u_x\, v_y + u_y\, v_x}{2} & \dfrac{u_x\, v_z + u_z\, v_x}{2} \\ \dfrac{u_x\, v_y + u_y\, v_x}{2} & \dfrac{2u_y\, v_y - u_x\, v_x - u_z\, v_z}{3} & \dfrac{u_y\, v_z + u_z\, v_y}{2} \\ \dfrac{u_x\, v_z + u_z\, v_x}{2} & \dfrac{u_y\, v_z + u_z\, v_y}{2} & \dfrac{2u_z\, v_z - u_x\, v_x - u_y\, v_y}{3} \end{pmatrix} .$$

As already anticipated, and with reference to Eq. (1.50), we can order the nonvanishing components of the antisymmetric tensor $\left.\mathbb{T}\right|_{\ell=1}$ into a vector, namely, the vector product of $\vec{u}$ and $\vec{v}$,

$$\vec{u} \times \vec{v} = \begin{pmatrix} u_y\, v_z - u_z\, v_y \\ u_z\, v_x - u_x\, v_z \\ u_x\, v_y - u_y\, v_x \end{pmatrix} , \qquad (\vec{u} \times \vec{v})_i = \epsilon_{ijk}\, u_j\, v_k \,, \tag{3.145}$$

where a summation over j and k is understood by the Einstein summation convention. The fact that $\left.\mathbb{T}\right|_{\ell=1}$ transforms as a (pseudo-)vector shows that it is possible to construct a vector whose components themselves are products of vector components. In Cartesian components, the "coupling coefficients" can directly be read off from Eq. (3.145), in the sense that the components of a tensor of rank one (of the vector product) are obtained by multiplying the products $u_j\, v_k$ of the components of the vectors $\vec{u}$ and $\vec{v}$ by the coupling coefficient ϵ_{ijk}.

However, the canonical formalism employs the spherical basis. We recall that the z component of the angular momentum operator has a particularly simple form in spherical coordinates,

$$L_z = -\mathrm{i}\left(x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial x}\right) = -\mathrm{i}\,\frac{\partial}{\partial \varphi} \,. \tag{3.146}$$

The spherical basis of vector components is chosen to generate an explicit φ dependence of the form $\exp(\mathrm{i}m\varphi)$, i.e., it consists of eigenfunctions of L_z. In the spherical basis, the components are denoted as

x_{+1}, x_0 and x_{-1}; they read as follows,

$$x_{+1} = -\frac{1}{\sqrt{2}}(x + \mathrm{i}\,y) = -\frac{1}{\sqrt{2}}|\vec{r}|\,\sin\theta\,\mathrm{e}^{\mathrm{i}\varphi} = \sqrt{\frac{4\pi}{3}}\,|\vec{r}|\,Y_{11}(\theta,\varphi)\,, \tag{3.147a}$$

$$x_0 = z = |\vec{r}|\,\cos\theta = \sqrt{\frac{4\pi}{3}}\,|\vec{r}|\,Y_{10}(\theta,\varphi)\,, \tag{3.147b}$$

$$x_{-1} = \frac{1}{\sqrt{2}}(x - \mathrm{i}\,y) = \frac{1}{\sqrt{2}}|\vec{r}|\,\sin\theta\,\mathrm{e}^{-\mathrm{i}\varphi} = \sqrt{\frac{4\pi}{3}}\,|\vec{r}|\,Y_{1-1}(\theta,\varphi)\,, \tag{3.147c}$$

where we have allowed ourselves the luxury to write $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$ explicitly. The emergence of phase factors of the form $\exp(\mathrm{i}\,m\,\varphi)$ with $m = -1, 0, 1$ is evident. The spherical components are complemented by spherical basis vectors as follows,

$$\vec{\mathrm{e}}_{+1} = -\frac{1}{\sqrt{2}}(\hat{e}_x + \mathrm{i}\,\hat{e}_y)\,, \qquad \vec{\mathrm{e}}_0 = \hat{e}_z\,, \qquad \vec{\mathrm{e}}_{-1} = \frac{1}{\sqrt{2}}(\hat{e}_x - \mathrm{i}\,\hat{e}_y)\,. \tag{3.148}$$

The coordinate vector can easily be expanded into the spherical basis,

$$\vec{r} = \sum_{q=-1}^{1} x_q\,\vec{\mathrm{e}}_q^{\,*} = \sum_{q=-1}^{1} (-1)^q\,x_{-q}\,\vec{\mathrm{e}}_q\,, \qquad \vec{\mathrm{e}}_q^{\,*} = (-1)^q\,\vec{\mathrm{e}}_{-q}\,. \tag{3.149}$$

The spherical basis vectors are normalized as follows,

$$\vec{\mathrm{e}}_q \cdot \vec{\mathrm{e}}_{q'} = (-1)^q\,\delta_{q\,-q'}\,, \qquad \vec{\mathrm{e}}_q \cdot \vec{\mathrm{e}}_{q'}^{\,*} = \delta_{q\,q'}\,. \tag{3.150}$$

The sum over q runs over the indices $q = -1, 0, 1$. For absolute clarity, we should mention that the latter Kronecker symbol in Eq. (3.149) is to be understood as $\delta_{q,-q'}$, i.e., q has to be equal to $-q'$ in order for the Kronecker symbol to be equal to unity rather than zero. Components can be extracted by calculating the scalar product of the coordinate vector $\vec{r}$ with a spherical basis vector,

$$\vec{r}\cdot\vec{\mathrm{e}}_q = \sum_{q'=-1}^{1} x_{q'}\,\vec{\mathrm{e}}_{q'}^{\,*}\cdot\vec{\mathrm{e}}_q = \sum_{q'=-1}^{1} x_{q'}\,\delta_{q'\,q} = x_q\,. \tag{3.151}$$

The paradigm of the vector coupling, or Clebsch–Gordan, coefficients, is that one obtains a tensor component of magnetic quantum number m for a tensor of rank j by coupling two tensors of rank j_1 and j_2 as follows,

$$w(jm) = \sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2} C^{jm}_{j_1 m_1 j_2 m_2}\,u(j_1\,m_1)\,v(j_2\,m_2)\,. \tag{3.152}$$

Here, $u(j_1\,m_1)$ and $v(j_2\,m_2)$ are two distinct vectors. In our example, we couple two tensors of rank one (the vectors $\vec{u}$ and $\vec{v}$ with spherical components u_{+1}, u_0 and u_{-1}, as well as v_{+1}, v_0 and v_{-1}), to a tensor of rank one, which is the vector product $\vec{w} = \vec{u}\times\vec{v}$. So, we have $j_1 = j_2 = j = 1$ for our example, which is given by Eq. (3.145) in Cartesian coordinates. The Clebsch–Gordan coefficients are tabulated and nowadays implemented in most modern computer algebra systems. Using tabulated values, one finds

$$w_q = \sum_{q'q''} C^{1q}_{1q'\,1q''}\,u_{q'}\,v_{q''}\,, \tag{3.153a}$$

$$w_{+1} = \frac{u_{+1}\,v_0 - u_0\,v_{+1}}{\sqrt{2}} = \left[\frac{\mathrm{i}}{\sqrt{2}}\right]\left\{-\frac{1}{\sqrt{2}}\left[(\vec{u}\times\vec{v})_x + \mathrm{i}\,(\vec{u}\times\vec{v})_y\right]\right\}\,, \tag{3.153b}$$

$$w_0 = \frac{u_{+1}\,v_{-1} - u_{-1}\,v_{+1}}{\sqrt{2}} = \left[\frac{\mathrm{i}}{\sqrt{2}}\right](\vec{u}\times\vec{v})_z\,, \tag{3.153c}$$

$$w_{-1} = \frac{u_0\,v_{-1} - u_{-1}\,v_0}{\sqrt{2}} = \left[\frac{\mathrm{i}}{\sqrt{2}}\right]\left\{\frac{1}{\sqrt{2}}\left[(\vec{u}\times\vec{v})_x - \mathrm{i}\,(\vec{u}\times\vec{v})_y\right]\right\}\,, \tag{3.153d}$$

where the subscripts x, y, z denote the Cartesian components of the vector product, i.e., $(u \times v)_z = u_x\, v_y - u_y\, v_x$ and further by cyclic permutation. A comparison of Eq. (3.145) with Eq. (3.147) shows that the components of $\vec{w}$ are equal to those obtained by writing $\vec{u} \times \vec{v}$ in the spherical basis, up to a prefactor $\mathrm{i}/\sqrt{2}$, i.e.,

$$\vec{w} = \sum_{q=-1}^{1} (-1)^q\, w_{-q}\, \vec{\mathrm{e}}_q = \frac{\mathrm{i}}{\sqrt{2}}\, (\vec{u} \times \vec{v}) \,. \tag{3.154}$$

One can form two further linear combinations of the spherical basis vectors $\vec{\mathrm{e}}_q$ which are of interest,

$$\vec{\mathrm{e}}_q \times \vec{\mathrm{e}}_{q'} = \mathrm{i}\,\sqrt{2} \sum_{\lambda=-1}^{1} C^{1\lambda}_{1q1q'}\, \vec{\mathrm{e}}_\lambda \,, \qquad \vec{r} = -\sqrt{3} \sum_{q=-1}^{1} \sum_{q'=-1}^{1} C^{00}_{1q1q'}\, x_q\, \vec{\mathrm{e}}_{q'} \,. \tag{3.155}$$

The first of these illustrates that the vector product of two basis vectors in the spherical basis again is a vector; the second clarifies that the coordinate vector $\vec{r}$ actually is a scalar under rotations, obtained as the scalar combination (tensor of rank zero, component number zero) composed out of the spherical coordinates x_q and the spherical basis vectors $\vec{\mathrm{e}}_q$. Of course, the vector composed of the components x_q is not a scalar. However, the scalar product of the vector composed of the x_q with the vector composed of the $\vec{\mathrm{e}}_q$ constitutes a physical vector $\vec{r}$ which does not change just upon a change of the reference frame [see Eq. (3.149)]. Upon rotation, this scalar product is equal to the coordinate vector obtained using the new coordinates x'_q, but multiplied with the rotated basis vectors $\vec{\mathrm{e}}'_q$, which leaves the physical coordinate vector $\vec{r}$ invariant. This corresponds to a passive interpretation of the rotation.

Let us now investigate the $(\ell = 2)$-component given in Eq. (3.144). It is symmetric and traceless and has five independent components. The counting works as follows: We have five independent components, because the matrix is symmetric and traceless. This leaves three off-diagonal and two diagonal components to be determined; the third component on the diagonal is fixed by the condition $\mathrm{trc}\,(\mathbb{T}|_{\ell=2}) = 0$. A scalar $(\ell = 0)$ has one component, a vector $(\ell = 1)$ has three magnetic components x_m (which depend on φ as $\exp(\mathrm{i}m\varphi)$ with $m = -1, 0, 1$), and a quadrupole tensor has five magnetic components [which depend on φ as $\exp(\mathrm{i}m\varphi)$ with $m = -2, -1, 0, 1, 2$]. The generalization calls for $2\ell + 1$ magnetic components for a tensor of rank ℓ. Again, using tabulated values for Clebsch–Gordan coefficients of the form $C^{2q}_{1q'\,1q''}$, with $q', q'' = -1, 0, 1$ and $q = -2, -1, 0, 1, 2$, we have

$$t_q = \sum_{q'q''} C^{2q}_{1q'\,1q''}\, u_{q'}\, v_{q''} \,, \tag{3.156a}$$

$$t_{+2} = u_{+1}\, v_{+1} \,, \qquad t_{+1} = \frac{u_{+1}\, v_0 + u_0\, v_{+1}}{\sqrt{2}} \,, \tag{3.156b}$$

$$t_0 = \frac{3\, u_0\, v_0 - \vec{u} \cdot \vec{v}}{\sqrt{6}} \,, \tag{3.156c}$$

$$t_{-1} = \frac{u_{-1}\, v_0 + u_0\, v_{-1}}{\sqrt{2}} \,, \qquad t_{-2} = u_{-1}\, v_{-1} \,, \tag{3.156d}$$

where the spherical components u_{-1}, u_0 and u_{+1} are defined in Eq. (3.147). If the operators $\vec{L}_1$ and $\vec{L}_2$ address the vectors $\vec{u}$ and $\vec{v}$ separately, then the functions t_q are eigenfunctions of the operator $\vec{L}^2 = (\vec{L}_1 + \vec{L}_2)^2$ with eigenvalue $\vec{L}^2 \to \ell(\ell + 1) = 6$ and of the z component $L_z = L_{1z} + L_{2z}$ with an eigenvalue q of L_z.

3.4.2 Vector Additions and Vector Spherical Harmonics

Vector spherical harmonics are obtained upon adding the "spin" of the photon, namely, the spherical basis vectors which are used in the expansion of the vector potential, to the spherical harmonics which represent the "orbital" angular momentum of the photon. Photons (light particles) are spin-1 objects. The spherical basis vectors are components of a tensor of rank one. To this tensor we add, vectorially, the orbital angular momentum ℓ of the photon, as manifest in the spherical harmonic. Hence, in view of Eq. (3.152), the vector spherical harmonic is given as

$$\vec{Y}^{\ell}_{j\mu}(\hat{r}) = \sum_{m=-\ell}^{\ell} \sum_{q=-1}^{1} C^{j\mu}_{\ell m\, 1q}\, Y_{\ell m}(\theta,\varphi)\, \vec{\mathrm{e}}_q \,. \tag{3.157}$$

Because neither the total angular momentum quantum number j nor the orbital angular momentum ℓ can be negative, the vector spherical harmonics with $j = -1$ and $\ell = -1$ vanish; this observation comes in handy in regard to a number of summations discussed in the following. From Eq. (3.148), we recall the basis vectors in the spherical basis,

$$\vec{\mathrm{e}}_{+1} = -\frac{1}{\sqrt{2}}\,(\hat{e}_x + \mathrm{i}\,\hat{e}_y)\,, \qquad \vec{\mathrm{e}}_0 = \hat{e}_z\,, \qquad \vec{\mathrm{e}}_{-1} = \frac{1}{\sqrt{2}}\,(\hat{e}_x - \mathrm{i}\,\hat{e}_y)\,. \tag{3.158}$$

The Clebsch–Gordan coefficients $C^{j\mu}_{\ell m\,1q}$ assemble a tensor of angular symmetry $j\mu$ from a spherical harmonic of angular symmertry ℓm and a spherical basis vector of angular symmetry $1q$. We prefer the above notation for the vector spherical harmonic; the magnetic projection μ may assume values from $-j$ to j. The superscript ℓ reminds us of the orbital momentum which was used in the construction of the vector spherical harmonic. The spin of the photon (equal to one) is added to the orbital angular momentum; hence the total angular momentum j can differ from ℓ by at most unity; otherwise the vector spherical harmonic vanishes.

The spin operators of the photon are given by the matrices $\mathbb{S}_i$ $(i = 1,2,3)$,

$$(\mathbb{S}_k)_{ij} = -\mathrm{i}\,\epsilon_{kij}\,, \tag{3.159}$$

where ϵ_{ijk} is the Levi–Cività tensor [see Eq. (1.50)]. The explicit representation for $k = 1,2,3$ reads as

$$\mathbb{S}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \qquad \mathbb{S}_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \qquad \mathbb{S}_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.160}$$

The matrix $\mathbb{M}$ given above is identified as the lower right 2×2 submatrix of $\mathbb{M}_1$, and also as the upper left 2×2 submatrix of $\mathbb{M}_3$. These matrices and the corresponding components of the angular momentum vector fulfill the algebraic relations

$$[\,\mathbb{S}_i, \mathbb{S}_j\,] = \mathrm{i}\,\epsilon_{ijk}\,\mathbb{S}_k\,, \tag{3.161}$$

where the Einstein summation convention is used for the sum over $k = 1,2,3$ on the right-hand side. The vector square of the $\mathbb{S}$ matrices is

$$\vec{\mathbb{S}}^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = S\,(S+1)\,\mathbb{1}_{3\times 3}\,, \qquad S = 1\,, \tag{3.162}$$

demonstrating that the photon is a spin-1 particle (with $S = 1$). The spherical basis vectors fulfill the relations [see Eq. (12.85) of U. D. Jentschura and G. S. Adkins, *Quantum Electrodynamics: Atoms, Lasers and Gravity* (World Scientific, Singapore, 2022)]

$$\mathbb{S}_z\,\vec{e}_\lambda = \mathbb{S}_3\,\vec{e}_\lambda = \lambda\,\vec{e}_\lambda\,, \qquad \lambda = -1,0,1\,. \tag{3.163}$$

The total angular momentum operator of the photon is given by

$$\vec{J} = \vec{L}\, \mathbb{1}_{3\times3} + \vec{\mathbb{S}}\,, \tag{3.164}$$

where we are pedantic in multiplying the orbital angular momentum operator by the three-dimensional unit matrix, implying that, say, the z component $L_z\,\vec{A} = L_z \mathbb{1}\cdot\vec{A}$ acts on the entire vector $\vec{A}$, i.e., on all of the components of $\vec{A}$ separately. The z component of $\vec{J}$ acts on a vector-valued function as $J_z\,\vec{V}(\theta,\varphi) = L_z\,\vec{V}(\theta,\varphi) + \mathbb{S}_z\cdot\vec{V}(\theta,\varphi)$. The vector spherical harmonics have the properties

$$\vec{J}^{\,2}\,\vec{Y}^{\ell}_{j\mu}(\theta,\varphi) = j(j+1)\,\vec{Y}^{\ell}_{j\mu}(\theta,\varphi)\,, \tag{3.165a}$$

$$\vec{L}^{\,2}\,\vec{Y}^{\ell}_{j\mu}(\theta,\varphi) = \ell(\ell+1)\,\vec{Y}^{\ell}_{j\mu}(\theta,\varphi)\,, \tag{3.165b}$$

$$J_z\,\vec{Y}^{\ell}_{j\mu}(\theta,\varphi) = \mu\,\vec{Y}^{\ell}_{j\mu}(\theta,\varphi)\,. \tag{3.165c}$$

The orthonormality relations are

$$\int \mathrm{d}\Omega\,\vec{Y}^{\ell\,*}_{j\mu}(\theta,\varphi)\cdot\vec{Y}^{\ell'}_{j'\mu'}(\theta,\varphi) = \delta_{jj'}\,\delta_{\ell\ell'}\,\delta_{\mu\mu'}\,, \tag{3.166a}$$

$$\sum_{\mu=-j}^{j}\int \mathrm{d}\Omega\,\vec{Y}^{\ell}_{j\mu}(\theta,\varphi)\otimes\vec{Y}^{\ell\,*}_{j\mu}(\theta,\varphi) = \frac{2j+1}{3}\,\mathbb{1}_{3\times3}\,, \tag{3.166b}$$

where we assume that the vector spherical harmonic is nonvanishing, i.e., $|j-\ell|\leq 1$. In the second sum, j and ℓ are held constant; they just have to be the same for both vector spherical harmonics but are not summed over. For given j, there are three possible values of ℓ, namely, $\ell = j-1, j, j+1$; summing over these, one obtains a factor $(2j+1)$ instead of $(2j+1)/3$ on the right-hand side.

There is also a completeness relation,

$$\sum_{j=\ell-1}^{\ell+1}\;\sum_{\mu=-j}^{j}\vec{Y}^{\ell}_{j\mu}(\theta,\varphi)\otimes\vec{Y}^{\ell *}_{j\mu}(\theta',\varphi') = \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\,Y^{*}_{\ell m}(\theta',\varphi')\;\mathbb{1}_{3\times3}\,, \tag{3.167}$$

which implies that

$$\sum_{\ell=0}^{\infty}\sum_{j=\ell-1}^{\ell+1}\;\sum_{\mu=-j}^{j}\vec{Y}^{\ell}_{j\mu}(\theta,\varphi)\otimes\vec{Y}^{\ell *}_{j\mu}(\theta',\varphi') = \frac{1}{\sin\theta}\,\delta(\theta-\theta')\,\delta(\varphi-\varphi')\;\mathbb{1}_{3\times3}\,. \tag{3.168}$$

Complex conjugation leads to the relation

$$\vec{Y}^{\ell\,*}_{j\mu}(\theta,\varphi) = (-1)^{\ell+1+j}\,(-1)^{\mu}\,\vec{Y}^{\ell}_{j\,-\mu}(\theta,\varphi)\,. \tag{3.169}$$

Explicit representations are given as follows,

$$\vec{Y}^{j}_{j\,\mu}(\theta,\varphi) = \frac{1}{\sqrt{j(j+1)}}\,\vec{L}Y_{j\mu}(\theta,\varphi)\,, \tag{3.170a}$$

$$\begin{aligned}\vec{Y}^{j-1}_{j\,\mu}(\theta,\varphi) &= \frac{1}{\sqrt{j\,(2j+1)}}\,\left(j\,\hat{r} + r\,\vec{\nabla}\right)\,Y_{j\mu}(\theta,\varphi)\\ &= -\frac{1}{\sqrt{j\,(2j+1)}}\,\left(\mathrm{i}\,\hat{r}\times\vec{L} - j\,\hat{r}\right)\,Y_{j\mu}(\theta,\varphi)\,,\end{aligned} \tag{3.170b}$$

$$\begin{aligned}\vec{Y}^{j+1}_{j\,\mu}(\theta,\varphi) &= -\frac{1}{\sqrt{(j+1)\,(2j+1)}}\,\left((j+1)\,\hat{r} - r\,\vec{\nabla}\right)\,Y_{j\mu}(\theta,\varphi)\\ &= -\frac{1}{\sqrt{(j+1)\,(2j+1)}}\,\left(\mathrm{i}\,\hat{r}\times\vec{L} + (j+1)\,\hat{r}\right)\,Y_{j\mu}(\theta,\varphi)\,.\end{aligned} \tag{3.170c}$$

Here, μ can take on the values $\mu = -j, \dots, j$. Again, we emphasize that vector spherical harmonics with $|j - \ell| > 1$ vanish.

The two equivalent representations of the vector spherical harmonics can reconciled with each other on the basis of the operator identity

$$(\mathrm{i}\,\hat{r} \times \vec{L})\, f(\theta,\varphi) = -r\, \vec{\nabla} f(\theta,\varphi)\,, \tag{3.171}$$

which is valid for any test function f that only depends on the angular variables.

We can write a representation of the vector spherical harmonics in terms of Clebsch-Gordan coefficients. For example, we have in the case $\ell = j$,

$$\vec{Y}^{j}_{j\mu}(\theta,\varphi) = C^{j\mu}_{j\,\mu-1\,1\,1}\, \vec{e}_{+1}\, Y_{j\,\mu-1}(\theta,\varphi) + C^{j\mu}_{j\mu\,1\,0}\, \vec{e}_0\, Y_{j\,\mu}(\theta,\varphi) + C^{j\mu}_{j\,\mu+1\,1\,-1}\, \vec{e}_{-1}\, Y_{j\,\mu+1}(\theta,\varphi)\,. \tag{3.172}$$

Using known formulas for the Clebsch–Gordan coefficients, the vector spherical harmonics with $\ell = j$ find the following representation,

$$\begin{aligned} \vec{Y}^{j}_{j\mu}(\theta,\varphi) = & -\sqrt{\frac{(j+\mu)\,(j-\mu+1)}{2j(j+1)}}\, Y_{j,\mu-1}(\theta,\varphi)\, \vec{e}_{+1} + \frac{\mu}{\sqrt{j(j+1)}}\, Y_{j,\mu}(\theta,\varphi)\, \vec{e}_0 \\ & + \sqrt{\frac{(j-\mu)\,(j+\mu+1)}{2j(j+1)}}\, Y_{j,\mu+1}(\theta,\varphi)\, \vec{e}_{-1}\,, \end{aligned} \tag{3.173a}$$

For the case $\ell = j - 1$, one has

$$\begin{aligned} \vec{Y}^{j-1}_{j\mu}(\theta,\varphi) = & \sqrt{\frac{(j+\mu-1)\,(j+\mu)}{2j(2j-1)}}\, Y_{j-1,\mu-1}(\theta,\varphi)\, \vec{e}_{+1} \\ & + \sqrt{\frac{(j-\mu)\,(j+\mu)}{j(2j-1)}}\, Y_{j-1,\mu}(\theta,\varphi)\, \vec{e}_0 \\ & + \sqrt{\frac{(j-\mu-1)\,(j-\mu)}{2j(2j-1)}}\, Y_{j-1,\mu+1}(\theta,\varphi)\, \vec{e}_{-1}\,, \end{aligned} \tag{3.173b}$$

Finally, for the case $\ell = j + 1$, one has

$$\begin{aligned} \vec{Y}^{j+1}_{j\mu}(\theta,\varphi) = & \sqrt{\frac{(j-\mu+1)\,(j-\mu+2)}{2(j+1)(2j+3)}}\, Y_{j+1,\mu-1}(\theta,\varphi)\, \vec{e}_{+1} \\ & - \sqrt{\frac{(j-\mu+1)\,(j+\mu+1)}{(j+1)(2j+3)}}\, Y_{j+1,\mu}(\theta,\varphi)\, \vec{e}_0 \\ & + \sqrt{\frac{(j+\mu+2)\,(j+\mu+1)}{2(j+1)(2j+3)}}\, Y_{j+1,\mu+1}(\theta,\varphi)\, \vec{e}_{-1}\,. \end{aligned} \tag{3.173c}$$

Here, one easily discerns the addition of the magnetic quantum number, of the spherical harmonic and the spherical basis vector, to the total magnetic projection of the vector spherical harmonic.

3.4.3 Scalar Helmholtz Green Function and Scalar Potential

We recall that the Helmholtz Green function is given as follows [see Eq. (3.13)],

$$G_R(k, \vec{r} - \vec{r}\,') = \frac{\exp(\mathrm{i}k\,|\vec{r} - \vec{r}\,'|)}{4\pi\epsilon_0\,|\vec{r} - \vec{r}\,'|}\,. \tag{3.174}$$

This Green function couples the scalar potential to the source, according to Eq. (3.94a),

$$\Phi(\vec{r},t) = \mathrm{e}^{-\mathrm{i}\,\omega\,t}\,\Phi_0(\vec{r})\,, \qquad \Phi_0(\vec{r}) = \int \mathrm{d}^3 r'\; G_R\left(\frac{\omega}{c}, \vec{r}-\vec{r}'\right)\,\rho_0(\vec{r}')\,. \tag{3.175}$$

For definiteness, we also recall the angular decomposition of the Green function for the Helmholtz equation according to Eq. (3.87),

$$G_R(k,\vec{r}-\vec{r}') = \frac{1}{\epsilon_0}\sum_{\ell,m} \mathrm{i}\,k\,j_\ell(k\,r_<)\,h^{(1)}_\ell(k\,r_>)\,Y_{\ell m}(\theta,\varphi)\,Y^*_{\ell m}(\hat{r}')\,. \tag{3.176}$$

Outside of the charge distribution, the scalar potential is thus given by

$$\Phi_0(\vec{r}) = \frac{\mathrm{i}\,k}{\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} p_{\ell m}\;h^{(1)}_\ell(k\,r)\;Y_{\ell m}(\theta,\varphi)\,, \tag{3.177}$$

with

$$p_{\ell m} = \int \mathrm{d}^3 r\; j_\ell(k\,r)\,\rho_0(\vec{r})\,Y^*_{\ell m}(\theta,\varphi) \approx \frac{k^\ell}{(2\ell+1)!!}\int \mathrm{d}^3 r\; r^\ell\,\rho_0(\vec{r})\,Y^*_{\ell m}(\theta,\varphi)\,. \tag{3.178}$$

The $p_{\ell m}$ generalize the multipole components of a static charge distribution for a dynamical process, namely, the emission of radiation. In the notation, we suppress their dependence on the wave number k. Indeed, for $k \to 0$ (zero-frequency radiation), the leading term in the expansion of $p_{\ell m}$ according to Eq. (3.178) is proportional to $q_{\ell m}$, a fact which is obvious from Eq. (3.71).

3.4.4 Tensor Helmholtz Green Function and Vector Potential

The tensor Helmholtz Green function is obtained from the Helmholtz Green function by a multiplication with the unit matrix,

$$\mathbb{G}_R(k,\vec{r}-\vec{r}') = \mathbb{1}_{3\times 3}\,G_R(k,\vec{r}-\vec{r}')\,. \tag{3.179}$$

It enters the equation that couples the vector potential to the source, Eq. (3.94b), which we write as follows,

$$\vec{A}(\vec{r},t) = \mathrm{e}^{-\mathrm{i}\omega t}\;\vec{A}_0(\vec{r})\,, \qquad \vec{A}_0(\vec{r}) = \int \mathrm{d}^3 r'\;\frac{1}{c^2}\,\mathbb{G}_R\left(\frac{\omega}{c}, \vec{r}-\vec{r}'\right)\cdot\vec{J}_0\left(\vec{r}'\right)\,. \tag{3.180}$$

Note the explicit matrix product of the tensor Green function and the current density, which differentiates Eq. (3.180) from (3.94b).

In order to proceed with the analysis of the tensor Green function, we first need to write a tensorial decomposition. The unit matrix in Eq. (3.179), which describes the spin of the photon, needs to be incorporated into the analysis. Of course, the hope is that once we form the tensor product of all the vector spherical harmonics pertaining to the same orbital angular momentum ℓ, we would somehow recover the angular structure in Eq. (3.87), namely, $\sum_m Y_{\ell m}(\theta,\varphi)\,Y^*_{\ell m}(\hat{r}')$, multiplied by the unit matrix $\mathbb{1}_{3\times 3}$. Indeed, we recall Eq. (3.167),

$$\sum_{j=\ell-1}^{\ell+1}\;\sum_{\mu=-j}^{j}\vec{Y}^{\ell}_{j\mu}(\theta,\varphi)\otimes\vec{Y}^{\ell *}_{j\mu}(\theta',\varphi') = \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\,Y^*_{\ell m}(\theta',\varphi')\;\mathbb{1}_{3\times 3}\,. \tag{3.181}$$

For the case $\ell = 0$, we recall the observation reported in the text following Eq. (3.157). Essentially, in Eq. (3.167), for given ℓ, one sums over the possible values of j, namely, $j = \ell-1, \ell, \ell+1$, and then over the possible magnetic projections μ, and obtains an angular structure which is familiar from Eq. (3.87). Here, the tensor product is the one which transforms the two vectors $Y_{\ell m}(\theta,\varphi)$ and $Y^*_{\ell m}(\theta',\varphi')$ into a matrix; pedantically, one might otherwise have indicated the transpose of the latter vector.

The angular decomposition of the tensor Green function (3.87) for the vector Helmholtz equation can thus be given in terms of the vector spherical harmonics,

$$
\begin{aligned}
\mathbb{G}_R(k,\vec{r}-\vec{r}\,') &= \frac{\exp\left(\mathrm{i}\,k\,|\vec{r}-\vec{r}\,'|\right)}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|}\;\mathbb{1}_{3\times 3} \\
&= \frac{\mathrm{i}\,k}{\epsilon_0}\sum_{j=0}^{\infty}\;\sum_{\mu=-j}^{j}\;\sum_{\ell=j-1}^{j+1}\; j_\ell(k\,r_<)\,h_\ell^{(1)}(k\,r_>)\,\vec{Y}_{j\mu}^{\ell}(\theta,\varphi)\otimes\vec{Y}_{j\mu}^{\ell*}(\theta',\varphi') \\
&= \frac{\mathrm{i}\,k}{\epsilon_0}\sum_{j=0}^{\infty}\;\sum_{\mu=-j}^{j}\;\Big(j_{j-1}(k\,r_<)\,h_{j-1}^{(1)}(k\,r_>)\,\vec{Y}_{j\mu}^{j-1}(\theta,\varphi)\otimes\vec{Y}_{j\mu}^{j-1*}(\theta',\varphi') \\
&\quad + j_j(k\,r_<)\,h_j^{(1)}(k\,r_>)\,\vec{Y}_{j\mu}^{j}(\theta,\varphi)\otimes\vec{Y}_{j\mu}^{j*}(\theta',\varphi') \\
&\quad + j_{j+1}(k\,r_<)\,h_{j+1}^{(1)}(k\,r_>)\,\vec{Y}_{j\mu}^{j+1}(\theta,\varphi)\otimes\vec{Y}_{j\mu}^{j+1*}(\theta',\varphi')\Big)\,. \qquad (3.182)
\end{aligned}
$$

So, outside of the charge distribution, the vector potential is thus given by

$$
\begin{aligned}
\vec{A}_0(\vec{r}) &= \frac{1}{c^2}\int \mathrm{d}^3r'\,\mathbb{G}_R(k,\vec{r}-\vec{r}\,')\cdot\vec{J}_0(\vec{r}\,') \\
&= \int \mathrm{d}^3r'\,\frac{\exp\left(\mathrm{i}\,k\,|\vec{r}-\vec{r}\,'|\right)}{4\pi\epsilon_0 c^2\,|\vec{r}-\vec{r}\,'|}\;\mathbb{1}_{3\times 3}\cdot\vec{J}_0(\vec{r}\,') \\
&= \mathrm{i}\,\mu_0\,k\sum_{j=0}^{\infty}\;\sum_{\mu=-j}^{j}\;\sum_{\ell=j-1}^{j+1}\; p_{j\mu}^{\ell}\;h_\ell^{(1)}(k\,r)\;\vec{Y}_{j\mu}^{\ell}(\theta,\varphi)\,, \qquad (3.183)
\end{aligned}
$$

with

$$
p_{j\mu}^{\ell} = \int \mathrm{d}^3r\,j_\ell(k\,r)\;\vec{J}_0(\vec{r})\cdot\vec{Y}_{j\mu}^{\ell*}(\theta,\varphi) \approx \frac{k^\ell}{(2\ell+1)!!}\int \mathrm{d}^3r\,r^\ell\;\vec{J}_0(\vec{r})\cdot\vec{Y}_{j\mu}^{\ell*}(\theta,\varphi)\,. \qquad (3.184)
$$

The advantage of Eq. (3.183) over Eq. (3.96) lies in the fact that the vector structure of the radiated vector potential is resolved and the spin of the photon is incorporated into the formalism.

But we are not quite there yet. Namely, the decomposition (3.183) does not clearly separate the longitudinal and transverse components to the electric and magnetic fields generated by the vector potential. An alternative decomposition, which accomplishes a separation into electric and magnetic multipole radiation, reads as follows,

$$
\begin{aligned}
\mathbb{G}_R(k,\vec{r}-\vec{r}\,') &= \frac{\mathrm{i}\,k}{\epsilon_0}\sum_{j=0}^{\infty}\sum_{\mu=-j}^{j}\Big(\vec{M}_{j\mu}^{(1)}(k,\vec{r}_>)\otimes\vec{M}_{j\mu}^{(0)*}(k,\vec{r}_<) \\
&\quad + \vec{N}_{j\mu}^{(1)}(k,\vec{r}_>)\otimes\vec{N}_{j\mu}^{(0)*}(k,\vec{r}_<) + \vec{L}_{j\mu}^{(1)}(k,\vec{r}_>)\otimes\vec{L}_{j\mu}^{(0)*}(k,\vec{r}_<)\Big)\,. \qquad (3.185)
\end{aligned}
$$

From the above discussion, it is clear that ℓ takes the role otherwise taken by j, because we have added the spin of the photon to its orbital angular momentum. We shall later see that the angular dependence of the functions $\vec{M}_{j\mu}^{(K)}$, $\vec{N}_{j\mu}^{(K)}$, and $\vec{L}_{j\mu}^{(K)}$, is given by vector spherical harmonics with a total angular momentum number j. The notation is to be explained in the following. For the magnetic (M), electric (N), and longitudinal (L) multipole moments, we have

$$
\vec{M}_{j\mu}^{(K)}(k,\vec{r}) = \frac{1}{\sqrt{j(j+1)}}\;f_j^{(K)}(k\,r)\;\vec{L}Y_{j\mu}(\theta,\varphi)\,, \qquad (3.186a)
$$

$$
\vec{N}_{j\mu}^{(K)}(k,\vec{r}) = \frac{\mathrm{i}}{k}\,\vec{\nabla}\times\vec{M}_{j\mu}^{(K)}(k,\vec{r})\,, \qquad (3.186b)
$$

$$
\vec{L}_{j\mu}^{(K)}(k,\vec{r}) = \frac{1}{k}\,\vec{\nabla}\left(f_j^{(K)}(k\,r)\,Y_{j\mu}(\theta,\varphi)\right)\,, \qquad (3.186c)
$$

and conversely

$$\vec{M}_{j\mu}^{(K)}(k,\vec{r}) = -\frac{\mathrm{i}}{k}\,\vec{\nabla}\times\vec{N}_{j\mu}^{(K)}(k,\vec{r})\,. \tag{3.187}$$

The definition of $\vec{M}_{00}^{(K)}(k,\vec{r})$ needs to be clarified for Eq. (3.186). In principle, we are dividing zero by zero because the application of the $\vec{L}$ operator to Y_{00} leads to zero, but the prefactor $1/\sqrt{j(j+1)}$ has a zero in the denominator. The solution is to allow for an infinitesimal displacement of the angular momentum $j \to j+\varepsilon$ before applying the $\vec{L}$ operator and then letting $\varepsilon \to 0$ at the end of the calculation. This clarifies that

$$\vec{M}_{00}^{(K)}(k,\vec{r}) = \vec{N}_{00}^{(K)}(k,\vec{r}) = \vec{0}\,. \tag{3.188}$$

The $f_j^{(K)}(k\,r)$ with $K=-1,0,1$ are Bessel and Hankel functions, as follows,

$$f_j^{(0)}(k\,r) = j_j(k\,r)\,, \qquad f_j^{(1)}(k\,r) = h_j^{(1)}(k\,r)\,, \qquad f_j^{(-1)}(k\,r) = h_j^{(2)}(k\,r)\,. \tag{3.189}$$

The vector multipole decomposition involves the following moments,

$$m_{j\mu} = \int \mathrm{d}^3r'\,\vec{J}_0(\vec{r}\,')\cdot\vec{M}_{j\mu}^{(0)*}(\theta',\varphi')\,, \tag{3.190a}$$

$$n_{j\mu} = \int \mathrm{d}^3r'\,\vec{J}_0(\vec{r}\,')\cdot\vec{N}_{j\mu}^{(0)*}(\theta',\varphi')\,, \tag{3.190b}$$

$$l_{j\mu} = \int \mathrm{d}^3r'\,\vec{J}_0(\vec{r}\,')\cdot\vec{L}_{j\mu}^{(0)*}(\theta',\varphi')\,, \tag{3.190c}$$

whose dependence on k is suppressed. (This dependence is due to the $f_j^{(K)}$ functions defined in Eq. (3.189), which enter the $\vec{M}_{j\mu}^{(0)*}$, $\vec{N}_{j\mu}^{(0)*}$, and $\vec{L}_{j\mu}^{(0)*}$, according to Eq. (3.186).) We note that the curl operator in Eq. (3.190b) admixes Bessel functions j_{j-1} and j_{j+1} to j_j. The approximation

$$f_j^{(0)}(k\,r) = j_j(k\,r) \approx \frac{(k\,r)^j}{(2j+1)!!} \tag{3.191}$$

is otherwise applicable for a localized source, in much the same way as in Eqs. (3.178) and (3.184). Finally, the decomposition of the vector potential reads as

$$\vec{A}_0(\vec{r}) = \mathrm{i}k\,\mu_0 \sum_{j=0}^{\infty}\sum_{\mu=-j}^{j}\left(m_{j\mu}\,\vec{M}_{j\mu}^{(1)}(k,\vec{r}) + n_{j\mu}\,\vec{N}_{j\mu}^{(1)}(k,\vec{r}) + l_{j\mu}\,\vec{L}_{j\mu}^{(1)}(k,\vec{r})\right)\,. \tag{3.192}$$

Here, the $m_{j\mu}$ are the magnetic multipole moments, the $n_{j\mu}$ are the electric multipole moments, and the $l_{j\mu}$ are longitudinal multipole moments which do not contribute to the fields.

A (perhaps) more systematic expansion writes the $\vec{M}_{j\mu}^{(K)}(k,\vec{r})$, $\vec{N}_{j\mu}^{(K)}(k,\vec{r})$, and $\vec{L}_{j\mu}^{(K)}(k,\vec{r})$ in terms of the vector spherical harmonics with $\ell = j-1, j, j+1$, but definite j,

$$\vec{M}_{j\mu}^{(K)}(k,\vec{r}) = f_j^{(K)}(k\,r)\,\vec{Y}_{j\mu}^{j}(\theta,\varphi)\,, \tag{3.193a}$$

$$\vec{N}_{j\mu}^{(K)}(k,\vec{r}) = -\sqrt{\frac{j+1}{2j+1}}\,f_{j-1}^{(K)}(k\,r)\,\vec{Y}_{j\,\mu}^{j-1}(\theta,\varphi) + \sqrt{\frac{j}{2j+1}}\,f_{j+1}^{(K)}(k\,r)\,\vec{Y}_{j\,\mu}^{j+1}(\theta,\varphi)\,, \tag{3.193b}$$

$$\vec{L}_{j\mu}^{(K)}(k,\vec{r}) = \sqrt{\frac{j}{2j+1}}\,f_{j-1}^{(K)}(k\,r)\,\vec{Y}_{j\,\mu}^{j-1}(\theta,\varphi) + \sqrt{\frac{j+1}{2j+1}}\,f_{j+1}^{(K)}(k\,r)\,\vec{Y}_{j\,\mu}^{j+1}(\theta,\varphi)\,. \tag{3.193c}$$

From Eq. (3.193b), one might think that $\vec{N}^{(K)}_{00}(k,\vec{r})$ could incur a nonvanishing contribution proportional to $\vec{Y}^{1}_{00}(\theta,\varphi)$, but the prefactor of this term vanishes. Equation (3.193c) teaches us that the term proportional to $\vec{Y}^{1}_{00}(\theta,\varphi)$ is part of the longitudinal component of the vector potential, which does not contribute to the electric and magnetic fields. There is no such thing as a photon with vanishing total angular momentum $j=\mu=0$.

We have just uncovered the fact that the only nanvanishing term with $j=\mu=0$ in the expansion of $A_0(\vec{r})$ according to Eq. (3.192) is the longitudinal component, proportional to $\vec{L}^{(1)}_{00}(k,\vec{r})$. For the magnetic and electric multipoles, proportional to $\vec{M}^{(1)}_{j\mu}(k,\vec{r})$ and $\vec{N}^{(1)}_{j\mu}(k,\vec{r})$, we might otherwise start the discussion at $j=1$. However, in order to ensure a certain uniformity of the notation, we will continue the summations with the term $j=\mu=0$ in the following.

Yet another representation is as follows,

$$\vec{M}^{(K)}_{j\mu}(k,\vec{r}) = f^{(K)}_j(k\,r)\,\vec{Y}^{j}_{j\mu}(\theta,\varphi)\,, \tag{3.194a}$$

$$\vec{N}^{(K)}_{j\mu}(k,\vec{r}) = \frac{1}{kr}\left\{\frac{\mathrm{d}[kr\,f^{(K)}_j(k\,r)]}{\mathrm{d}(kr)}[\mathrm{i}\hat{r}\times\vec{Y}^{j}_{j\mu}(\theta,\varphi)] - \hat{r}\sqrt{j(j+1)}f^{(K)}_j(k\,r)\,Y_{j\mu}(\theta,\varphi)\right\}, \tag{3.194b}$$

$$\vec{L}^{(K)}_{j\mu}(k,\vec{r}) = \hat{r}\,\frac{\mathrm{d}[f^{(K)}_j(k\,r)]}{\mathrm{d}(k\,r)}\,Y_{j\mu}(\theta,\varphi) - \sqrt{j(j+1)}\,\frac{1}{k\,r}\,f^{(K)}_j(k\,r)\,(\mathrm{i}\hat{r}\times\vec{Y}^{j}_{j\mu}(\theta,\varphi))\,. \tag{3.194c}$$

Let us now try to interpret the contributions $\vec{M}^{(K)}_{j\mu}(k,\vec{r})$, $\vec{N}^{(K)}_{j\mu}(k,\vec{r})$, and $\vec{L}^{(K)}_{j\mu}(k,\vec{r})$ physically. From Eq. (3.186c), we infer that $\vec{L}^{(K)}_{j\mu}(k,\vec{r})$ is the "longitudinal" solution constructed by taking the gradient of a scalar solution of the Helmholtz equation. The magnetic and electric fields are obtained from the vector potential, via the equations

$$\vec{B}_0(\vec{r}) = \vec{\nabla}\times\vec{A}_0(\vec{r})\,, \qquad \vec{E}_0(\vec{r}) = \frac{\mathrm{i}c^2}{\omega}\,\vec{\nabla}\times\vec{B}_0(\vec{r})\,, \tag{3.195}$$

in the source-free region. Because $\vec{L}^{(K)}_{j\mu}(k,\vec{r})$ is a gradient of a scalar solution of the Helmholtz equation, its curl vanishes. Furthermore, in view of Eq. (3.195), the fields generated by the longitudinal components of the vector potential vanish. We note that in view of the relation $\vec{E}(\vec{r},t) = -\vec{\nabla}\Phi(\vec{r},t) - \partial_t\vec{A}(\vec{r},t)$, the electric field can alternatively be calculated as

$$\vec{E}_0(\vec{r}) = -\vec{\nabla}\Phi_0(\vec{r}) + \mathrm{i}\omega\vec{A}_0(\vec{r})\,. \tag{3.196}$$

We note that $\vec{L}^{(K)}_{j\mu}(k,\vec{r})$ is longitudinal, just like $\vec{\nabla}\Phi_0(\vec{r})$ (finally, it is just a gradient). We shall thus later have to show that in calculating $\vec{E}_0(\vec{r})$, the gradient of Φ_0 given in Eq. (3.177) actually cancels against the time derivative of the longitudinal contribution proportional to $\mathrm{i}\omega\,\vec{L}^{(K)}_{j\mu}(k,\vec{r})$ from the time derivative of the vector potential.

Hence, we have

$$\vec{E}_0(\vec{r}) = \mathrm{i}\omega\vec{A}_{0\perp}(\vec{r})\,, \tag{3.197a}$$

$$\vec{A}_{0\perp}(\vec{r}) = \mathrm{i}k\,\mu_0\sum_{j=0}^{\infty}\sum_{\mu=-j}^{j}\left(m_{j\mu}\,\vec{M}^{(1)}_{j\mu}(k,\vec{r}) + n_{j\mu}\,\vec{N}^{(1)}_{j\mu}(k,\vec{r})\right)\,, \tag{3.197b}$$

$$\vec{E}_0(\vec{r}) = -\,k^2\,c\,\mu_0\sum_{j=0}^{\infty}\sum_{\mu=-j}^{j}\left(m_{j\mu}\,\vec{M}^{(1)}_{j\mu}(k,\vec{r}) + n_{j\mu}\,\vec{N}^{(1)}_{j\mu}(k,\vec{r})\right)\,. \tag{3.197c}$$

From Eq. (3.186a), we infer that $\vec{M}_{j\mu}$ is the (normalized) elementary solution consisting of a Bessel function times $\vec{L}\,Y_{j\mu}(\theta,\varphi) \propto \vec{Y}^{j}_{j\mu}(\theta,\varphi)$. It is (by construction) purely transverse, because $\hat{r}\cdot\vec{Y}^{j}_{j\mu}(\theta,\varphi) \propto \vec{r}\cdot(\vec{r}\times\vec{\nabla})Y_{j\mu}(\theta,\varphi) = 0$. The general rationale is this: The $\vec{M}$ terms are the magnetic multipoles. We have $\vec{B} = \vec{\nabla}\times\vec{A}$ and $\vec{E} \propto \vec{\nabla}\times\vec{B}$. The electric field $\vec{E} \propto \vec{M}$ in this case is transverse, i.e., $\hat{r}\cdot\vec{E} = 0$. This is the right characteristic of a magnetic multipole.

In view of Eq. (3.186b), we infer that $\vec{N}_{j\mu}$ is the solution constructed by the taking the curl of $\vec{M}_{j\mu}$. The $\vec{N}_{j\mu}$ terms are the electric multipoles. Forming the curl of $\vec{N}$, one obtains an expression for $\vec{B}$ which is proportional to $\vec{M}_{j\mu}$, which evidently is transverse. In this particular case, one has $\vec{r}\cdot\vec{B} = 0$. The general result is as follows,

$$\vec{B}_0(\vec{r}) = k^2\,\mu_0 \sum_{j=0}^{\infty}\sum_{\mu=-j}^{j} \left(m_{j\mu}\,\vec{N}^{(1)}_{j\mu}(k,\vec{r}) - n_{j\mu}\,\vec{M}^{(1)}_{j\mu}(k,\vec{r}) \right) , \tag{3.197d}$$

as will be shown below. A transverse magnetic field is characteristic of an electric multipole. One observes, for the electric dipole, that the exact expression for the magnetic field, given in Eq. (3.136), is transverse, while the exact expression for the electric field, given in Eq. (3.137), is not transverse.

We summarize,

$$\text{Magnetic Multipole:} \quad \hat{r}\cdot\vec{B}_0(\vec{r}) \neq 0\,, \qquad \hat{r}\cdot\vec{E}_0(\vec{r}) = 0\,, \tag{3.198}$$

$$\text{Electric Multipole:} \quad \hat{r}\cdot\vec{E}_0(\vec{r}) \neq 0\,, \qquad \hat{r}\cdot\vec{B}_0(\vec{r}) = 0\,, \tag{3.199}$$

for radiation emanating from a localized source.

It is perhaps instructive to discuss the "construction principle" that leads to Eq. (3.192), i.e., the rationale behind the transformation from Eq. (3.183) to (3.192). For given j and μ, one first identifies the maximal longitudinal subcomponent and calls it $\vec{L}^{(1)}_{j\mu}(k,\vec{r})$. One then constructs the maximal transverse component whose scalar product with $\hat{r}$ vanishes, and identifies it as $\vec{M}^{(1)}_{j\mu}(k,\vec{r}) = 0$. The rest of the component of the vector potential with given j and μ finally is the electric multipole, called $\vec{N}^{(1)}_{j\mu}(k,\vec{r}) = 0$.

A remark is in order. For the magnetic multipoles, the corresponding term in $\vec{A}_0(\vec{r})$ [the term containing $\vec{M}^{(1)}_{j\mu}(k,\vec{r})$] is transverse, i.e., its scalar product with $\hat{r}$ vanishes. The electric field is parallel to the time derivative of the sum of the non-longitudinal components of $\vec{A}_0(\vec{r})$. The electric field involves the combination $m_{j\mu}\,\vec{M}^{(1)}_{j\mu}(k,\vec{r}) + n_{j\mu}\,\vec{N}^{(1)}_{j\mu}(k,\vec{r})$, so it is transverse for the magnetic multipoles. The magnetic induction field has the combination $m_{j\mu}\,\vec{N}^{(1)}_{j\mu}(k,\vec{r}) - n_{j\mu}\,\vec{M}^{(1)}_{j\mu}(k,\vec{r})$, so it is transverse for the electric multipoles. This observation generalizes the behavior found in Eqs. (3.136) and (3.137), for the exact expressions pertaining to the electric and magnetic fields radiated by a dipole (the magnetic field was found to be transverse).

3.5 Radiation and Angular Momenta

3.5.1 Radiated Electric and Magnetic Fields

We shall now try to verify Eq. (3.197d) explicitly. The magnetic radiated field is calculated as follows,

$$\begin{aligned}\vec{B}_0(\vec{r}) &= \vec{\nabla} \times \vec{A}_0(\vec{r}) \\ &= \mathrm{i}\, k\, \mu_0 \sum_{\ell m} \left(m_{j\mu}\, \vec{\nabla} \times \vec{M}^{(1)}_{j\mu}(k,\vec{r}) + n_{j\mu}\, \vec{\nabla} \times \vec{N}^{(1)}_{j\mu}(k,\vec{r}) + l_{j\mu}\, \vec{\nabla} \times \vec{L}^{(1)}_{j\mu}(k,\vec{r}) \right) \\ &= \mathrm{i}\, k\, \mu_0 \sum_{j\mu} \left(m_{j\mu} \left(-\mathrm{i}\, k\, \vec{N}^{(1)}_{j\mu}(k,\vec{r}) \right) + n_{j\mu} \left(\mathrm{i}\, k\, \vec{M}^{(1)}_{j\mu}(k,\vec{r}) \right) \right) \\ &= k^2\, \mu_0 \sum_{j\mu} \left(m_{j\mu}\, \vec{N}^{(1)}_{j\mu}(k,\vec{r}) - n_{j\mu}\, \vec{M}^{(1)}_{j\mu}(k,\vec{r}) \right) , \qquad (3.200)\end{aligned}$$

where we have used Eqs. (3.186) and (3.187). In order to calculate the electric field, one observes that in a source-free region, the Ampere–Maxwell law implies that [see Eq. (1.179)]

$$\vec{\nabla} \times \vec{B}_0(\vec{r}) = -\frac{\mathrm{i}\omega}{c^2}\, \vec{E}_0(\vec{r}) \qquad \Rightarrow \qquad \vec{E}_0(\vec{r}) = \frac{\mathrm{i}c^2}{\omega}\, \vec{\nabla} \times \vec{B}_0(\vec{r})\,. \qquad (3.201)$$

Using Eqs. (3.186b) and (3.187), this is evaluated as

$$\begin{aligned}\vec{E}_0(\vec{r}) &= \frac{\mathrm{i}c^2}{\omega} \vec{\nabla} \times \vec{B}_0(\vec{r}) \\ &= k^2\, \mu_0 \left(\frac{\mathrm{i}c^2}{\omega} \right) \sum_{j\mu} \left(m_{j\mu}\, \vec{\nabla} \times \vec{N}^{(1)}_{j\mu}(k,\vec{r}) - n_{j\mu}\, \vec{\nabla} \times \vec{M}^{(1)}_{j\mu}(k,\vec{r}) \right) \\ &= \frac{\mathrm{i}\, k^2\, \mu_0\, c^2}{\omega} \sum_{j\mu} \left(m_{j\mu} \left(\mathrm{i}\, k\, \vec{M}^{(1)}_{j\mu}(k,\vec{r}) \right) - n_{j\mu} \left(-\mathrm{i}\, k\, \vec{N}^{(1)}_{j\mu}(k,\vec{r}) \right) \right) \\ &= \frac{(\mathrm{i}k)\, \mathrm{i}\, k^2\, \mu_0\, c^2}{c\, k} \sum_{j\mu} \left(m_{j\mu}\, \vec{M}^{(1)}_{j\mu}(k,\vec{r}) + n_{j\mu}\, \vec{N}^{(1)}_{j\mu}(k,\vec{r}) \right) \\ &= -\, k^2\, c\, \mu_0 \sum_{j\mu} \left(m_{j\mu}\, \vec{M}^{(1)}_{j\mu}(k,\vec{r}) + n_{j\mu}\, \vec{N}^{(1)}_{j\mu}(k,\vec{r}) \right) , \qquad (3.202)\end{aligned}$$

confirming the result anticipated in Eq. (3.197c).

3.5.2 Gauge Condition, Vector and Scalar Potentials

We have defined the multipoles for the radiated scalar potential in Eq. (3.178); the longitudinal components of the vector potential have been discussed in Eqs. (3.190c) and (3.186c). In our discussion of the radiated electric field, we anticipated the cancellation of the longitudinal contributions, in between the gradient of the scalar potential and the time derivative of the vector potential, in Eqs. (3.196) and (3.197). In consequence, we anticipate a simple relation between the $p_{\ell m}$ coefficients from Eq. (3.178) and the $l_{j\mu}$ coefficients from Eq. (3.190c). The derivation of this relation is necessary in order to establish the fulfillment of the Lorenz gauge condition and will be discussed in the following. We remember that the equations used by us in order to relate the sources to the potentials, namely, Eqs. (3.175) and (3.180), precisely are the Lorenz gauge versions, equivalent to Eq. (1.92).

In the derivation, we shall use the representation (3.186c) for the $\vec{L}^{(K)}_{j\mu}(k,\vec{r})$ functions. Furthermore, we have the charge conservation condition $\vec{\nabla}\cdot\vec{J}(\vec{r},t) = -\partial_t \rho(\vec{r},t)$ and so $\vec{\nabla}\cdot\vec{J}_0(\vec{r}) = \mathrm{i}\omega\rho_0(\vec{r})$. A simple partial integration then leads to the formulas

$$
\begin{aligned}
l_{j\mu} &= \int \mathrm{d}^3 r'\, \vec{J}_0(\vec{r}\,') \cdot \vec{L}^{(0)*}_{j\mu}(\theta',\varphi') \\
&= \int \mathrm{d}^3 r'\, \vec{J}_0(\vec{r}\,') \cdot \vec{\nabla}' \left(\frac{1}{k}\, j_j(k\,r')\, Y^*_{j\mu}(\theta',\varphi') \right) \\
&= -\int \mathrm{d}^3 r' \left[\vec{\nabla}' \cdot \vec{J}_0(\vec{r}\,') \right] \frac{1}{k}\, j_j(k\,r')\, Y^*_{j\mu}(\theta',\varphi') \\
&= -\int \mathrm{d}^3 r'\, \mathrm{i}\,\omega\, \rho_0(\vec{r}\,')\, \frac{1}{k}\, j_j(k\,r')\, Y^*_{j\mu}(\theta',\varphi') \\
&= -\mathrm{i}\frac{\omega}{k} \int \mathrm{d}^3 r'\, \rho_0(\vec{r}\,')\, j_j(k\,r')\, Y^*_{j\mu}(\theta',\varphi') = -\mathrm{i}\, c\, p_{j\mu}\,.
\end{aligned}
\tag{3.203}
$$

With $p_{j\mu} = \int \mathrm{d}^3 r'\, \rho_0(\vec{r}\,')\, j_j(k\,r')\, Y^*_{j\mu}(\theta',\varphi')$, we thus verify the relation

$$
l_{j\mu} = -\mathrm{i} c\, p_{j\mu}\,. \tag{3.204}
$$

Let us recall the gauge condition (1.84) fulfilled by the potentials,

$$
\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0\,. \tag{3.205}
$$

In view of the decompositions (3.177) and (3.192), we have for the scalar and vector potentials,

$$
\Phi_0(\vec{r}) = \frac{\mathrm{i}k}{\epsilon_0} \sum_{j=0}^{\infty} \sum_{\mu=-j}^{j} p_{j\mu}\, h^{(1)}_j(k\,r)\, Y_{j\mu}(\theta,\varphi)\,, \tag{3.206}
$$

and

$$
\vec{A}_0(\vec{r}) = \mathrm{i}k\,\mu_0 \sum_{j=0}^{\infty} \sum_{\mu=-j}^{j} \left(m_{j\mu}\, \vec{M}^{(1)}_{j\mu}(k,\vec{r}) + n_{j\mu}\, \vec{N}^{(1)}_{j\mu}(k,\vec{r}) + l_{j\mu}\, \vec{L}^{(1)}_{j\mu}(k,\vec{r}) \right)\,. \tag{3.207}
$$

In the mixed frequency-coordinate representation, the Lorentz gauge condition reads as

$$
\vec{\nabla}\cdot\vec{A}_0(\vec{r}) - \frac{\mathrm{i}\omega}{c^2}\Phi_0(\vec{r}) = 0\,. \tag{3.208}
$$

Because the magnetic and electric multipole terms are divergence-free [see Eq. (3.197b)], the Lorenz gauge condition is equivalent to

$$
\mathrm{i}k \sum_{j\mu} \left\{ \mu_0\, l_{j\mu}\, \vec{\nabla}\cdot\vec{L}^{(1)}_{j\mu}(k,\vec{r}) + \frac{(-\mathrm{i}\,\omega)}{\epsilon_0\, c^2}\, p_{j\mu}\, h^{(1)}_j(k\,r)\, Y_{j\mu}(\theta,\varphi) \right\} = 0\,, \tag{3.209}
$$

which implies that

$$
\mathrm{i}k \sum_{j\mu} \left\{ l_{j\mu}\, \vec{\nabla}\cdot\vec{L}^{(1)}_{j\mu}(k,\vec{r}) - \mathrm{i}\, kc\, p_{j\mu}\, h^{(1)}_j(k\,r)\, Y_{j\mu}(\theta,\varphi) \right\} = 0\,. \tag{3.210}
$$

We use Eq. (3.186c), in the form

$$
\vec{L}^{(1)}_{j\mu}(k,\vec{r}) = \frac{1}{k}\,\vec{\nabla}\left(h^{(1)}_j(k\,r)\, Y_{j\mu}(\theta,\varphi) \right)\,, \tag{3.211}
$$

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{L}^{(1)}_{j\mu}(k,\vec{r}) &= \frac{1}{k}\,\vec{\nabla}^2\left(h^{(1)}_j(k\,r)\, Y_{j\mu}(\theta,\varphi) \right) \\
&= -k\, h^{(1)}_j(k\,r)\, Y_{j\mu}(\theta,\varphi)\,.
\end{aligned}
\tag{3.212}
$$

Advantage has been taken of the fact that the function $h_j^{(1)}(k\,r)\,Y_{j\mu}(\theta,\varphi)$ fulfills the Helmholtz equation, i.e., $(\vec\nabla^2+k^2)h_j^{(1)}(k\,r)\,Y_{j\mu}(\theta,\varphi)=0$. Inserting Eq. (3.211) into (3.210), we see that the gauge condition is equivalent to the relation

$$\sum_{j\mu}(l_{j\mu}+\mathrm{i}\,c\,p_{j\mu})\;h_{j\mu}^{(1)}(k\,r)\,Y_{j\mu}(\theta,\varphi)=0\,, \tag{3.213}$$

which is fulfilled in view of Eq. (3.204).

Using Eq. (3.204), we should now be able to verify the cancellation of the longitudinal contributions to the electric field, in the steps leading from Eq. (3.196) to (3.197a). To this end, we calculate

$$\begin{aligned}
\vec E_0(\vec r) = &-\vec\nabla\Phi_0(\vec r)+\mathrm{i}\omega\vec A_0(\vec r)\\
= &-\vec\nabla\left(\frac{\mathrm{i}k}{\epsilon_0}\sum_{j\mu}p_{j\mu}\,h_j^{(1)}(kr)\,Y_{j\mu}(\hat r)\right)\\
&+\mathrm{i}\,\omega\left(\mathrm{i}k\mu_0\sum_{j\mu}\left\{m_{j\mu}\,\vec M_{j\mu}^{(1)}(k,\vec r)+n_{j\mu}\,\vec N_{j\mu}^{(1)}(k,\vec r)+l_{j\mu}\,\vec L_{j\mu}^{(1)}(k,\vec r)\right\}\right)\\
= &-k^2\mu_0\,c\sum_{j\mu}\left\{m_{j\mu}\,\vec M_{j\mu}^{(1)}(k,\vec r)+n_{j\mu}\,\vec N_{j\mu}^{(1)}(k,\vec r)\right\}+\vec{\mathcal E}(\vec r)\,,
\end{aligned} \tag{3.214}$$

where the first term is the result given in Eq. (3.197c) and the additional term $\vec{\mathcal E}(\vec r)$ must be shown to vanish,

$$\begin{aligned}
\vec{\mathcal E}(\vec r) = &-\frac{\mathrm{i}k}{\epsilon_0}\sum_{j\mu}\left\{p_{j\mu}\,\vec\nabla\left(h_j^{(1)}(kr)\,Y_{j\mu}(\hat r)\right)+\left(-\frac{\epsilon_0}{\mathrm{i}k}\right)\,\mathrm{i}\omega(\mathrm{i}k\mu_0)l_{j\mu}\,\vec L_{j\mu}^{(1)}(k,\vec r)\right\}\\
= &-\frac{\mathrm{i}k}{\epsilon_0}\sum_{j\mu}\left\{p_{j\mu}\,\vec\nabla\left(h_j^{(1)}(kr)\,Y_{j\mu}(\hat r)\right)-\mathrm{i}kc\,\mu_0\,\epsilon_0\,l_{j\mu}\,\vec L_{j\mu}^{(1)}(k,\vec r)\right\}\\
= &-\frac{\mathrm{i}k^2}{\epsilon_0}\sum_{j\mu}\left\{\left(p_{j\mu}-\frac{\mathrm{i}}{c}\,l_{j\mu}\right)\;\vec L_{j\mu}^{(1)}(k,\vec r)\right\}=0\,,
\end{aligned} \tag{3.215}$$

again in view of Eq. (3.204). We have shown that the longitudinal components of the electric field cancel, as they should, in the source-free region. A few remarks are in order. The factor $[-\epsilon_0/(\mathrm{i}k)]$ in the first line is simply introduced in order to cancel the first prefactor. Alternatively, one may observe that the extra term in the electric field, from the longitudinal components of the vector potential, simply reads as $-k^2\,c\,\mu_0\,\sum_{j\mu}l_{j\mu}\,\vec L_{j\mu}^{(1)}(k,\vec r)$, with the prefactor being fixed in Eq. (3.202).

3.5.3 Different Representations for the Vector Spherical Harmonics

In the following, we shall endeavor in rather difficult calculations. It is a good moment to rest and to write down that most important representations of the vector spherical harmonics, and of the multipole functions.

We start from Eqs. (3.170a) and (3.173a),

$$
\begin{aligned}
\vec{Y}^{j}_{j\,\mu}(\theta,\varphi) &= \frac{1}{\sqrt{j(j+1)}}\,\vec{L}Y_{j\mu}(\theta,\varphi)\,,\\
&= -\sqrt{\frac{(j+\mu)\,(j-\mu+1)}{2j(j+1)}}\,Y_{j,\mu-1}(\theta,\varphi)\,\vec{\mathrm{e}}_{+1} + \frac{\mu}{\sqrt{j(j+1)}}\,Y_{j,\mu}(\theta,\varphi)\,\vec{\mathrm{e}}_{0}\\
&\quad + \sqrt{\frac{(j-\mu)\,(j+\mu+1)}{2j(j+1)}}\,Y_{j,\mu+1}(\theta,\varphi)\,\vec{\mathrm{e}}_{-1}\,.
\end{aligned}
\tag{3.216a}
$$

Equations (3.170b) and (3.173b) state that

$$
\begin{aligned}
\vec{Y}^{j-1}_{j\,\mu}(\theta,\varphi) &= \frac{1}{\sqrt{j\,(2j+1)}}\,\left(j\,\hat{r} + r\,\vec{\nabla}\right)\,Y_{j\mu}(\theta,\varphi)\\
&= -\frac{1}{\sqrt{j\,(2j+1)}}\,\left(\mathrm{i}\,\hat{r}\times\vec{L} - j\,\hat{r}\right)\,Y_{j\mu}(\theta,\varphi)\,,\\
&= \sqrt{\frac{(j+\mu-1)\,(j+\mu)}{2j(2j-1)}}\,Y_{j-1,\mu-1}(\theta,\varphi)\,\vec{\mathrm{e}}_{+1} + \sqrt{\frac{(j-\mu)\,(j+\mu)}{j(2j-1)}}\,Y_{j-1,\mu}(\theta,\varphi)\,\vec{\mathrm{e}}_{0}\\
&\quad + \sqrt{\frac{(j-\mu-1)\,(j-\mu)}{2j(2j-1)}}\,Y_{j-1,\mu+1}(\theta,\varphi)\,\vec{\mathrm{e}}_{-1}\,,
\end{aligned}
\tag{3.216b}
$$

Finally, we have from Eq. (3.170c) and (3.173c),

$$
\begin{aligned}
\vec{Y}^{j+1}_{j\,\mu}(\theta,\varphi) &= -\frac{1}{\sqrt{(j+1)\,(2j+1)}}\,\left((j+1)\,\hat{r} - r\,\vec{\nabla}\right)\,Y_{j\mu}(\theta,\varphi)\\
&= -\frac{1}{\sqrt{(j+1)\,(2j+1)}}\,\left(\mathrm{i}\,\hat{r}\times\vec{L} + (j+1)\,\hat{r}\right)\,Y_{j\mu}(\theta,\varphi)\\
&= \sqrt{\frac{(j-\mu+1)\,(j-\mu+2)}{2(j+1)(2j+3)}}\,Y_{j+1,\mu-1}(\theta,\varphi)\,\vec{\mathrm{e}}_{+1} - \sqrt{\frac{(j-\mu+1)\,(j+\mu+1)}{(j+1)(2j+3)}}\,Y_{j+1,\mu}(\theta,\varphi)\,\vec{\mathrm{e}}_{0}\\
&\quad + \sqrt{\frac{(j+\mu+2)\,(j+\mu+1)}{2(j+1)(2j+3)}}\,Y_{j+1,\mu+1}(\theta,\varphi)\,\vec{\mathrm{e}}_{-1}\,.
\end{aligned}
\tag{3.216c}
$$

The spherical harmonics are given by Eq. (3.217),

$$
Y_{j\,\mu}(\theta,\varphi) = (-1)^{\mu}\left(\frac{2j+1}{4\pi}\,\frac{(j-\mu)!}{(j+\mu)!}\right)^{1/2}\,P_j^{|\mu|}(\cos\theta)\,\mathrm{e}^{\mathrm{i}\mu\varphi}\,.
\tag{3.217}
$$

We also summarize Eqs. (3.186), (3.193), and (3.194) as follows,

$$\begin{aligned}\vec{M}^{(K)}_{j\mu}(k,\vec{r}) &= \frac{1}{\sqrt{j(j+1)}}\,\vec{L}\left(f^{(K)}_j(k\,r)\,Y_{j\mu}(\theta,\varphi)\right)\\ &= f^{(K)}_j(k\,r)\,\vec{Y}^j_{j\mu}(\theta,\varphi) = -\frac{\mathrm{i}}{k}\,\vec{\nabla}\times\vec{N}^{(K)}_{j\mu}(k,\vec{r})\,,\end{aligned}\tag{3.218a}$$

$$\begin{aligned}\vec{N}^{(K)}_{j\mu}(k,\vec{r}) &= \frac{\mathrm{i}}{k}\,\vec{\nabla}\times\vec{M}^{(K)}_{j\mu}(k,\vec{r})\,,\\ &= -\sqrt{\frac{j+1}{2j+1}}\,f^{(K)}_{j-1}(k\,r)\,\vec{Y}^{j-1}_{j\,\mu}(\theta,\varphi) + \sqrt{\frac{j}{2j+1}}\,f^{(K)}_{j+1}(k\,r)\,\vec{Y}^{j+1}_{j\,\mu}(\theta,\varphi)\,,\\ &= \frac{1}{kr}\left\{\frac{\mathrm{d}[kr\,f^{(K)}_j(k\,r)]}{\mathrm{d}(kr)}[\mathrm{i}\hat{r}\times\vec{Y}^j_{j\mu}(\theta,\varphi)] - \hat{r}\sqrt{j(j+1)}f^{(K)}_j(k\,r)\,Y_{j\mu}(\theta,\varphi)\right\},\end{aligned}\tag{3.218b}$$

$$\begin{aligned}\vec{L}^{(K)}_{j\mu}(k,\vec{r}) &= \frac{1}{k}\,\vec{\nabla}\left(f^{(K)}_j(k\,r)\,Y_{j\mu}(\theta,\varphi)\right)\\ &= \sqrt{\frac{j}{2j+1}}\,f^{(K)}_{j-1}(k\,r)\,\vec{Y}^{j-1}_{j\,\mu}(\theta,\varphi) + \sqrt{\frac{j+1}{2j+1}}\,f^{(K)}_{j+1}(k\,r)\,\vec{Y}^{j+1}_{j\,\mu}(\theta,\varphi)\\ &= \hat{r}\,\frac{\mathrm{d}[f^{(K)}_j(k\,r)]}{\mathrm{d}(k\,r)}\,Y_{j\mu}(\theta,\varphi) - \sqrt{j(j+1)}\,\frac{1}{k\,r}\,f^{(K)}_j(k\,r)\,(\mathrm{i}\hat{r}\times\vec{Y}^j_{j\mu}(\theta,\varphi))\,.\end{aligned}\tag{3.218c}$$

These results will be needed in the analysis of the Poynting vector.

3.5.4 Poynting Vector

We start from the relations (3.197c) and (3.197d),

$$\vec{E}_0(\vec{r}) = -k^2\,c\,\mu_0\sum_{j=0}^{\infty}\sum_{\mu=-j}^{j}\left(m_{j\mu}\,\vec{M}^{(1)}_{j\mu}(k,\vec{r}) + n_{j\mu}\,\vec{N}^{(1)}_{j\mu}(k,\vec{r})\right)\,,\tag{3.219}$$

and

$$\vec{B}_0(\vec{r}) = k^2\,\mu_0\sum_{j=0}^{\infty}\sum_{\mu=-j}^{j}\left(m_{j\mu}\,\vec{N}^{(1)}_{j\mu}(k,\vec{r}) - n_{j\mu}\,\vec{M}^{(1)}_{j\mu}(k,\vec{r})\right)\,.\tag{3.220}$$

The Poynting vector, which carries the physical dimension of radiated power per area, is

$$\vec{S}(\vec{r},t) = \frac{1}{\mu_0}\,\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\,,\qquad \vec{S}_0(\vec{r}) = \frac{1}{2\mu_0}\,\vec{E}_0(\vec{r})\times\vec{B}^*_0(\vec{r})\,.\tag{3.221}$$

From Eq. (3.186a), we recall the relation

$$\vec{M}^{(K)}_{j\mu}(k,\vec{r}) = f^{(K)}_j(k\,r)\,\vec{Y}^j_{j\mu}(\theta,\varphi)\,,\tag{3.222}$$

which implies that

$$\vec{M}^{(1)}_{j\mu}(k,\vec{r}) = f^{(1)}_j(k\,r)\,\vec{Y}^j_{j\mu}(\theta,\varphi) = h^{(1)}_j(k\,r)\,\vec{Y}^j_{j\mu}(\theta,\varphi)\,.\tag{3.223}$$

In view of the asymptotics (for large $k\,r$)

$$h^{(1)}_j(k\,r)\to\frac{\mathrm{e}^{\mathrm{i}\,k\,r-(j+1)\mathrm{i}\pi/2}}{k\,r}\,,\tag{3.224}$$

we have

$$\vec{M}^{(1)}_{j\mu}(k,\vec{r}) = h^{(1)}_j(k\,r)\,\vec{Y}^j_{j\mu}(\theta,\varphi) \to \frac{\mathrm{e}^{\mathrm{i}\,k\,r-(j+1)\mathrm{i}\pi/2}}{k\,r}\,\vec{Y}^j_{j\mu}(\theta,\varphi) = (-\mathrm{i})^{j+1}\,\frac{\mathrm{e}^{\mathrm{i}\,k\,r}}{k\,r}\,\vec{Y}^j_{j\mu}(\theta,\varphi)\,. \tag{3.225}$$

From Eq. (3.186b), we infer that

$$\vec{N}^{(K)}_{j\mu}(k,\vec{r}) = -\sqrt{\frac{j+1}{2j+1}}\,f^{(K)}_{j-1}(k\,r)\,\vec{Y}^{j-1}_{j\,\mu}(\theta,\varphi) + \sqrt{\frac{j}{2j+1}}\,f^{(K)}_{j+1}(k\,r)\,\vec{Y}^{j+1}_{j\,\mu}(\theta,\varphi)\,, \tag{3.226}$$

This implies that for large r,

$$\begin{aligned}\vec{N}^{(1)}_{j\mu}(k,\vec{r}) =&\ -\sqrt{\frac{j+1}{2j+1}}\,h^{(1)}_{j-1}(k\,r)\,\vec{Y}^{j-1}_{j\,\mu}(\theta,\varphi) + \sqrt{\frac{j}{2j+1}}\,h^{(1)}_{j+1}(k\,r)\,\vec{Y}^{j+1}_{j\,\mu}(\theta,\varphi)\\ &\to\ -\sqrt{\frac{j+1}{2j+1}}\,(-\mathrm{i})^{(j-1)+1}\,\frac{\mathrm{e}^{\mathrm{i}\,k\,r}}{k\,r}\,\vec{Y}^{j-1}_{j\,\mu}(\theta,\varphi) + \sqrt{\frac{j}{2j+1}}\,(-\mathrm{i})^{(j+1)+1}\,\frac{\mathrm{e}^{\mathrm{i}\,k\,r}}{k\,r}\,\vec{Y}^{j+1}_{j\,\mu}(\theta,\varphi)\\ &\to\ -(-\mathrm{i})^j\,\frac{\mathrm{e}^{\mathrm{i}\,k\,r}}{k\,r}\,\left(\sqrt{\frac{j+1}{2j+1}}\,\vec{Y}^{j-1}_{j\,\mu}(\theta,\varphi) + \sqrt{\frac{j}{2j+1}}\,\vec{Y}^{j+1}_{j\,\mu}(\theta,\varphi)\right)\,. \end{aligned}\tag{3.227}$$

We use the fact that $h^{(1)}_{j-1}$ and $h^{(1)}_{j+1}$ only differ by a phase $(-\mathrm{i})^2 = -1$ in the long-distance limit. From Eqs. (3.170b) and (3.170c), one may derive the relation

$$\sqrt{\frac{j+1}{2j+1}}\,\vec{Y}^{j-1}_{j\,\mu}(\theta,\varphi) + \sqrt{\frac{j}{2j+1}}\,\vec{Y}^{j+1}_{j\,\mu}(\theta,\varphi) = -\mathrm{i}\,\left(\hat{r}\times\vec{Y}^j_{j\,\mu}(\theta,\varphi)\right)\,. \tag{3.228}$$

The long-range asymptotics of $\vec{N}^{(1)}_{j\mu}(k,\vec{r})$ are thus given by

$$\begin{aligned}\vec{N}^{(1)}_{j\mu}(k,\vec{r}) =&\ -(-\mathrm{i})^j\,\frac{\mathrm{e}^{\mathrm{i}\,k\,r}}{k\,r}\,\left(\sqrt{\frac{j+1}{2j+1}}\,\vec{Y}^{j-1}_{j\,\mu}(\theta,\varphi) + \sqrt{\frac{j}{2j+1}}\,\vec{Y}^{j+1}_{j\,\mu}(\theta,\varphi)\right)\\ =&\ -(-\mathrm{i})^j\,\frac{\mathrm{e}^{\mathrm{i}\,k\,r}}{k\,r}\,(-\mathrm{i})\,\left(\hat{r}\times\vec{Y}^j_{j\,\mu}(\theta,\varphi)\right) = -(-\mathrm{i})^{j+1}\,\frac{\mathrm{e}^{\mathrm{i}\,k\,r}}{k\,r}\,\left(\hat{r}\times\vec{Y}^j_{j\,\mu}(\theta,\varphi)\right)\,. \end{aligned}\tag{3.229}$$

We now use Eqs. (3.197c) and (3.197d) and the long-range asymptotic formulas,

$$\vec{M}^{(1)}_{j\mu}(k,\vec{r}) \to (-\mathrm{i})^{j+1}\,\frac{\mathrm{e}^{\mathrm{i}\,k\,r}}{k\,r}\,\vec{Y}^j_{j\mu}(\theta,\varphi)\,, \qquad \vec{N}^{(1)}_{j\mu}(k,\vec{r}) \to -(-\mathrm{i})^{j+1}\,\frac{\mathrm{e}^{\mathrm{i}\,k\,r}}{k\,r}\,\left(\hat{r}\times\vec{Y}^j_{j\,\mu}(\theta,\varphi)\right)\,. \tag{3.230}$$

Hence,

$$\begin{aligned}\vec{E}_0(\vec{r}) \to&\ -k^2\,c\,\mu_0\sum_{j=0}^{\infty}\sum_{\mu=-j}^{j}\left(m_{j\mu}\,(-\mathrm{i})^{j+1}\,\frac{\mathrm{e}^{\mathrm{i}\,k\,r}}{k\,r}\,\vec{Y}^j_{j\mu}(\theta,\varphi) - n_{j\mu}\,\left[(-\mathrm{i})^{j+1}\,\frac{\mathrm{e}^{\mathrm{i}\,k\,r}}{k\,r}\,\left(\hat{r}\times\vec{Y}^j_{j\,\mu}(\theta,\varphi)\right)\right]\right)\,,\\ =&\ -k^2\,c\,\mu_0\,\frac{\mathrm{e}^{\mathrm{i}\,k\,r}}{k\,r}\,\sum_{j=0}^{\infty}\sum_{\mu=-j}^{j}(-\mathrm{i})^{j+1}\,\left(m_{j\mu}\,\vec{Y}^j_{j\mu}(\theta,\varphi) - n_{j\mu}\,\left(\hat{r}\times\vec{Y}^j_{j\,\mu}(\theta,\varphi)\right)\right)\,, \end{aligned}\tag{3.231}$$

and

$$\begin{aligned}\vec{B}_0(\vec{r}) =&\ k^2\,\mu_0\sum_{j=0}^{\infty}\sum_{\mu=-j}^{j}\left(m_{j\mu}\,\left[-(-\mathrm{i})^{j+1}\,\frac{\mathrm{e}^{\mathrm{i}\,k\,r}}{k\,r}\,\left(\hat{r}\times\vec{Y}^j_{j\,\mu}(\theta,\varphi)\right)\right] - n_{j\mu}\,(-\mathrm{i})^{j+1}\,\frac{\mathrm{e}^{\mathrm{i}\,k\,r}}{k\,r}\,\vec{Y}^j_{j\mu}(\theta,\varphi)\right)\,,\\ =&\ -k^2\,\mu_0\,\frac{\mathrm{e}^{\mathrm{i}\,k\,r}}{k\,r}\,\sum_{j=0}^{\infty}\sum_{\mu=-j}^{j}(-\mathrm{i})^{j+1}\,\left(m_{j\mu}\,\left(\hat{r}\times\vec{Y}^j_{j\,\mu}(\theta,\varphi)\right) + n_{j\mu}\,\vec{Y}^j_{j\mu}(\theta,\varphi)\right)\,,\\ \vec{B}^*_0(\vec{r}) =&\ -k^2\,\mu_0\,\frac{\mathrm{e}^{-\mathrm{i}\,k\,r}}{k\,r}\,\sum_{j'=0}^{\infty}\sum_{\mu'=-j'}^{j'}\mathrm{i}^{j'+1}\,\left(m^*_{j'\mu'}\,\left(\hat{r}\times\vec{Y}^{j'\,*}_{j'\,\mu'}(\theta,\varphi)\right) + n^*_{j'\mu'}\,\vec{Y}^{j'\,*}_{j'\mu'}(\theta,\varphi)\right)\,. \end{aligned}\tag{3.232}$$

The Poynting vector evaluates to

$$\vec{E}_0(\vec{r}) = -k^2\, c\, \mu_0\, \frac{\mathrm{e}^{\mathrm{i}\,k\,r}}{k\,r} \sum_{j=0}^{\infty} \sum_{\mu=-j}^{j} (-\mathrm{i})^{j+1} \left(m_{j\mu}\, \vec{Y}^{j}_{j\mu}(\theta,\varphi) - n_{j\mu} \left(\hat{r} \times \vec{Y}^{j}_{j\,\mu}(\theta,\varphi) \right) \right) , \tag{3.233}$$

$$\vec{B}^*_0(\vec{r}) = -k^2\, \mu_0\, \frac{\mathrm{e}^{-\mathrm{i}\,k\,r}}{k\,r} \sum_{j'=0}^{\infty} \sum_{\mu'=-j'}^{j'} \mathrm{i}^{j'+1} \left(m^*_{j'\mu'} \left(\hat{r} \times \vec{Y}^{j'\,*}_{j'\,\mu'}(\theta,\varphi) \right) + n^*_{j'\mu'}\, \vec{Y}^{j'\,*}_{j'\mu'}(\theta,\varphi) \right) , \tag{3.234}$$

$$\begin{aligned}
\vec{S}_0(\vec{r}) &= \frac{1}{2\mu_0}\, \vec{E}_0(\vec{r}) \times \vec{B}^*_0(\vec{r}) \\
&= \frac{1}{2}\, \frac{k^4\, c\, \mu_0}{(k\,r)^2} \sum_{j=0}^{\infty} \sum_{\mu=-j}^{j} (-\mathrm{i})^{j+1} \left(m_{j\mu}\, \vec{Y}^{j}_{j\mu}(\theta,\varphi) - n_{j\mu} \left(\hat{r} \times \vec{Y}^{j}_{j\,\mu}(\theta,\varphi) \right) \right) \\
&\quad \times \sum_{j'=0}^{\infty} \sum_{\mu'=-j'}^{j'} \mathrm{i}^{j'+1} \left(m^*_{j'\mu'} \left(\hat{r} \times \vec{Y}^{j'\,*}_{j'\,\mu'}(\theta,\varphi) \right) + n^*_{j'\mu'}\, \vec{Y}^{j'\,*}_{j'\mu'}(\theta,\varphi) \right) \\
&= \frac{k^2\, c\, \mu_0}{2\, r^2} \sum_{j=0}^{\infty} \sum_{\mu=-j}^{j} \sum_{j'=0}^{\infty} \sum_{\mu'=-j'}^{j'} \mathrm{i}^{j'-j} \left(m_{j\mu}\, \vec{Y}^{j}_{j\mu}(\theta,\varphi) - n_{j\mu} \left(\hat{r} \times \vec{Y}^{j}_{j\,\mu}(\theta,\varphi) \right) \right) \\
&\quad \times \left(m^*_{j'\mu'} \left(\hat{r} \times \vec{Y}^{j'\,*}_{j'\,\mu'}(\theta,\varphi) \right) + n^*_{j'\mu'}\, \vec{Y}^{j'\,*}_{j'\mu'}(\theta,\varphi) \right) ,
\end{aligned} \tag{3.235}$$

which is a double-infinite sum. The total power radiated through a sphere of radius r is

$$P = \int \mathrm{d}\Omega\, r^2\, \hat{r} \cdot \vec{S}_0(\vec{r}) \,, \tag{3.236}$$

and considerable simplifications occur in its calculation. Here,

$$\begin{aligned}
P &= \int \mathrm{d}\Omega\, r^2\, \hat{r} \cdot \vec{S}_0(\vec{r}) = T_1 + T_2 \,, \\
T_1 &= \frac{k^2\, c\, \mu_0}{2} \int \mathrm{d}\Omega\, \hat{r} \cdot \sum_{jj'\mu\mu'} \mathrm{i}^{j'-j} \left(m_{j\mu}\, m^*_{j'\mu'}\, \vec{Y}^{j}_{j\mu}(\theta,\varphi) \times \left(\hat{r} \times \vec{Y}^{j'\,*}_{j'\,\mu'}(\theta,\varphi) \right) \right. \\
&\quad \left. - n_{j\mu}\, n^*_{j'\mu'} \left(\hat{r} \times \vec{Y}^{j}_{j\,\mu}(\theta,\varphi) \right) \times \vec{Y}^{j'\,*}_{j'\mu'}(\theta,\varphi) \right) , \\
T_2 &= \frac{k^2\, c\, \mu_0}{2} \int \mathrm{d}\Omega\, \hat{r} \cdot \sum_{jj'\mu\mu'} \mathrm{i}^{j'-j} \left(m_{j\mu}\, n^*_{j'\mu'}\, \vec{Y}^{j}_{j\mu}(\theta,\varphi) \times \vec{Y}^{j'\,*}_{j'\,\mu'}(\theta,\varphi) \right. \\
&\quad \left. - n_{j\mu}\, m^*_{j'\mu'} \left(\hat{r} \times \vec{Y}^{j}_{j\,\mu}(\theta,\varphi) \right) \times \left(\hat{r} \times \vec{Y}^{j'\,*}_{j'\,\mu'}(\theta,\varphi) \right) \right) .
\end{aligned} \tag{3.237}$$

Now the angular integrals have to be evaluated. We recall Eq. (3.170a),

$$\vec{Y}^{j}_{j\,\mu}(\theta,\varphi) = \frac{1}{\sqrt{j(j+1)}}\, \vec{L} Y_{j\mu}(\theta,\varphi) \,, \qquad \hat{r} \cdot \vec{Y}^{j}_{j\,\mu}(\theta,\varphi) = \frac{1}{\sqrt{j(j+1)}}\, \hat{r} \cdot \vec{L} Y_{j\mu}(\theta,\varphi) = 0 \,, \tag{3.238}$$

where $\vec{L} = -\mathrm{i}\,(\vec{r} \times \vec{\nabla})$. Hence,

$$
\begin{aligned}
&\int \mathrm{d}\Omega\, \hat{r} \cdot \vec{Y}^{j}_{j\mu}(\theta,\varphi) \times \left(\hat{r} \times \vec{Y}^{j'\,*}_{j'\,\mu'}(\theta,\varphi)\right) \\
&\quad = \int \mathrm{d}\Omega\, \hat{r} \cdot \left[\hat{r}\, \left(\vec{Y}^{j}_{j\mu}(\theta,\varphi) \cdot \vec{Y}^{j'\,*}_{j'\,\mu'}(\theta,\varphi)\right) - \vec{Y}^{j'\,*}_{j'\,\mu'}(\theta,\varphi)\, \left(\hat{r} \cdot \vec{Y}^{j}_{j\mu}(\theta,\varphi)\right)\right] \\
&\quad = \int \mathrm{d}\Omega\, \left[\vec{Y}^{j}_{j\mu}(\theta,\varphi) \cdot \vec{Y}^{j'\,*}_{j'\,\mu'}(\theta,\varphi) - \left(\hat{r} \cdot \vec{Y}^{j'\,*}_{j'\,\mu'}(\theta,\varphi)\right)\, \left(\hat{r} \cdot \vec{Y}^{j}_{j\mu}(\theta,\varphi)\right)\right] \\
&\quad = \int \mathrm{d}\Omega\, \vec{Y}^{j}_{j\mu}(\theta,\varphi) \cdot \vec{Y}^{j'\,*}_{j'\,\mu'}(\theta,\varphi) = \delta_{jj'}\, \delta_{\mu\mu'}\,, \qquad (3.239)
\end{aligned}
$$

where we have used Eq. (3.238). Similarly, one shows that

$$
\int \mathrm{d}\Omega\, \hat{r} \cdot \left[\left(\hat{r} \times \vec{Y}^{j}_{j\,\mu}(\theta,\varphi)\right) \times \vec{Y}^{j'\,*}_{j'\mu'}(\theta,\varphi)\right] = -\delta_{jj'}\, \delta_{\mu\mu'}\,. \qquad (3.240)
$$

The two identities

$$
\int \mathrm{d}\Omega\, \hat{r} \cdot \left(\vec{Y}^{j}_{j\mu}(\theta,\varphi) \times \vec{Y}^{j'\,*}_{j'\,\mu'}(\theta,\varphi)\right) = 0\,, \qquad (3.241)
$$

$$
\int \mathrm{d}\Omega\, \hat{r} \cdot \left(\hat{r} \times \vec{Y}^{j}_{j\,\mu}(\theta,\varphi)\right) \times \left(\hat{r} \times \vec{Y}^{j'\,*}_{j'\,\mu'}(\theta,\varphi)\right) = 0\,, \qquad (3.242)
$$

are a little harder to show and are left as exercises. Armed with the results (3.239)—(3.242), we can show that $T_2 = 0$ and simplify Eq. (3.233) drastically,

$$
P = \frac{k^2\, c\, \mu_0}{2} \sum_{j\mu} \left(|m_{j\mu}|^2 + |n_{j\mu}|^2\right)\,. \qquad (3.243)
$$

3.5.5 Half–Wave Antenna

Let us carry out the multipole decomposition explicitly for an example problem. We consider a half-wave antenna with a current inside the antenna,

$$
I = I_0\, \cos\left(\frac{2\pi z}{\lambda}\right)\, \mathrm{e}^{-\mathrm{i}\omega t}\,, \qquad (3.244)
$$

where $-\lambda/4 < z < \lambda/4$. The length of the antenna therefore is exactly equal to half the wavelength, hence the name. In order to invoke our formalism, we first need to translate the current into a current density. A first guess might be

$$
\vec{J}(\vec{r}) = \vec{J}_0(\vec{r})\, \mathrm{e}^{-\mathrm{i}\omega t}\,, \qquad \vec{J}_0(\vec{r}) = \hat{\mathrm{e}}_z\, I_0\, \cos\left(\frac{2\pi z}{\lambda}\right)\, \delta(x)\, \delta(y)\,, \qquad (3.245)
$$

because the antenna is oriented parallel to the z axis. However, this expressions leads to problems as we put in the formulas $x = r\, \sin\theta\, \cos\varphi$ and $y = r\, \sin\theta\, \sin\varphi$ for the coordinates. How are we supposed to evaluate the integrals over θ and φ if they enter the arguments of the Dirac-δ function in such an awkward way? We therefore use the apparent radial symmetry of the problem, with respect to the z axis, and write

$$
\vec{J}_0(r,\theta,\varphi) = \hat{\mathrm{e}}_z\, I_0\, \cos\left(\frac{2\pi r}{\lambda}\right)\, [2\delta(\theta) + 2\delta(\pi-\theta)]\; \Theta\left(\frac{\lambda}{4} - r\right)\, \frac{1}{2\pi\, r^2\, \sin(\theta)} \qquad (3.246)
$$

We use the convention for the book: [see Eq. **dirac_half**],

$$\int_a^b \mathrm{d}x\, \delta(x-a)\, f(x) = \frac{1}{2}\, f(a)\,, \qquad \int_a^b \mathrm{d}x\, \delta(x-b)\, f(x) = \frac{1}{2}\, f(b)\,, \tag{3.247}$$

for a Dirac-δ functions "on the boundary". One may thus calculate the integral over the (partial) surface area of a sphere, centered at the origin, which encompasses the antenna. The infinitesimal area element is

$$\mathrm{d}A = R^2\, \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi\,. \tag{3.248}$$

The surface integral, with the polar angle $\theta \in (0, \epsilon)$, samples points whose z coordinate is just $z = R$, i.e., points in the vicinity of the "pole" of the "sampling sphere". It evaluates to

$$\begin{aligned}\int \mathrm{d}A\, \vec{J}_0(R,\theta,\varphi) &= R^2 \int_0^\epsilon \mathrm{d}\,\theta \sin\theta \int_0^{2\pi} \mathrm{d}\varphi\, \vec{J}_0(R,\theta,\varphi) \\ &= \hat{\mathrm{e}}_z\, I_0\, \cos\left(\frac{2\pi R}{\lambda}\right) \int_0^{2\pi} \mathrm{d}\varphi \int_0^\epsilon \mathrm{d}\,\theta \sin\theta\, R^2\, \left[2\delta(\theta)\right]\, \frac{1}{2\pi\, R^2\, \sin(\theta)} \\ &= \hat{\mathrm{e}}_z\, I_0\, \cos\left(\frac{2\pi R}{\lambda}\right) \int_0^\epsilon \mathrm{d}\,\theta\, \left[2\delta(\theta)\right] = \hat{\mathrm{e}}_z\, I_0\, \cos\left(\frac{2\pi z}{\lambda}\right)\,.\end{aligned} \tag{3.249}$$

Assembling the current density from terms near the "North" and the "South" pole, one has

$$\vec{J}_0(r,\theta,\varphi) = I_0\, \cos\left(\frac{2\pi r}{\lambda}\right) \frac{\delta(\theta) + \delta(\pi-\theta)}{\pi\, r^2\, \sin(\theta)}\, \Theta(\tfrac{1}{4}\lambda - r)\, \hat{\mathrm{e}}_z\,. \tag{3.250}$$

In order to calculate the magnetic multipoles using Eq. (3.186a), one recalls that according to Eq. (3.170a),

$$\vec{Y}^{j}_{j\,\mu}(\hat{r}) = \frac{1}{\sqrt{j(j+1)}}\, \vec{L}\, Y_{j\mu}(\hat{r})\,, \tag{3.251}$$

and so

$$\hat{e}_z \cdot \vec{Y}^{j}_{j\,\mu}(\hat{r}) = \frac{1}{\sqrt{j(j+1)}}\, L_z Y_{j\mu}(\hat{r}) = \frac{1}{\sqrt{j(j+1)}}\, \mu\, Y_{j\mu}(\hat{r})\,. \tag{3.252}$$

Thus,

$$\begin{aligned} m_{j\mu} &= \int \mathrm{d}^3 r\, \vec{J}_0(\vec{r}) \cdot \vec{M}^{(0)*}_{j\mu}(\vec{r}) = \int \mathrm{d}^3 r\, j_0(k\,r)\, \vec{J}_0(\vec{r}) \cdot \vec{Y}^{j*}_{j\mu}(\theta,\varphi) \\ &= \int \mathrm{d}\varphi \int \mathrm{d}\theta\, \sin\theta \int_0^{\lambda/4} \mathrm{d}r\, r^2\, j_0(k\,r)\, I_0\, \cos(k\,r)\, \frac{\delta(\theta)+\delta(\pi-\theta)}{\pi\, r^2\, \sin(\theta)}\, \hat{\mathrm{e}}_z \cdot \vec{Y}^{j*}_{j\mu}(\theta,\varphi) \\ &= \frac{I_0}{2\pi} \int \mathrm{d}\varphi \int_0^{\lambda/4} \mathrm{d}r\, j_0(k\,r)\, \cos(k\,r)\, \left[\hat{\mathrm{e}}_z \cdot \vec{Y}^{j*}_{j\mu}(0,\varphi) + \hat{\mathrm{e}}_z \cdot \vec{Y}^{j*}_{j\mu}(\pi,\varphi)\right] \\ &= \frac{I_0}{2\pi} \int \mathrm{d}\varphi \int_0^{\lambda/4} \mathrm{d}r\, j_0(k\,r)\, \cos(k\,r)\, \frac{1}{\sqrt{j\,(j+1)}}\, \left[\mu\, Y^*_{j\mu}(0,\varphi) + \mu\, Y^*_{j\mu}(\pi,\varphi)\right]\,. \end{aligned} \tag{3.253}$$

From Eq. (3.217), one has

$$\begin{aligned} Y_{j\mu}(0,\varphi) &= \delta_{\mu 0}\, \sqrt{\frac{2j+1}{4\pi}}\,, \\ Y_{j\mu}(\pi,\varphi) &= (-1)^j\, \delta_{\mu 0}\, \sqrt{\frac{2j+1}{4\pi}}\,. \end{aligned} \tag{3.254}$$

Both results are independent of φ, as they should be on the pole caps of the unit sphere. The latter result follows from the former by the parity transformation $\theta \to \pi - \theta$, and $\varphi \to \varphi + \pi$ (parity transformation). In view of $\mu\,\delta_{\mu 0} = 0$, it follows that

$$m_{j\mu} = 0\,, \tag{3.255}$$

or in other words, all the magnetic multipoles vanish.

Now we turn our attention to the electric multipoles. From Eq. (3.173b), we have

$$\begin{aligned}
\hat{\mathrm{e}}_z \cdot \vec{Y}^{j-1\,*}_{j\,\mu}(\theta,\varphi) &= \sqrt{\frac{(j+\mu-1)\,(j+\mu)}{2j(2j-1)}}\,Y_{j-1,\mu-1}(\theta,\varphi)\,\hat{\mathrm{e}}_z\cdot\vec{\mathrm{e}}^{\,*}_{+1} + \sqrt{\frac{(j-\mu)\,(j+\mu)}{j(2j-1)}}\,Y_{j-1,\mu}(\theta,\varphi)\,\hat{\mathrm{e}}_z\cdot\vec{\mathrm{e}}^{\,*}_{0} \\
&\quad + \sqrt{\frac{(j-\mu-1)\,(j-\mu)}{2j(2j-1)}}\,Y_{j-1,\mu+1}(\theta,\varphi)\,\hat{\mathrm{e}}_z\cdot\vec{\mathrm{e}}^{\,*}_{-1} \\
&= \sqrt{\frac{(j-\mu)\,(j+\mu)}{j(2j-1)}}\,Y_{j-1,\mu}(\theta,\varphi)\,,
\end{aligned} \tag{3.256}$$

because $\vec{\mathrm{e}}_q\,\vec{\mathrm{e}}^{\,*}_{q'} = \delta_{q\,q'}$. At the "North" pole of the unit sphere, one therefore has the relation,

$$\begin{aligned}
\hat{\mathrm{e}}_z \cdot \vec{Y}^{j-1\,*}_{j\,\mu}(0,\varphi) &= \sqrt{\frac{(j-\mu)\,(j+\mu)}{j(2j-1)}}\,Y_{j-1,\mu}(0,\varphi) = \sqrt{\frac{(j-\mu)\,(j+\mu)}{j(2j-1)}}\,\delta_{\mu 0}\sqrt{\frac{2(j-1)+1}{4\pi}} \\
&= \delta_{\mu 0}\sqrt{\frac{j^2}{j(2j-1)}}\sqrt{\frac{2j-1}{4\pi}} = \delta_{\mu 0}\sqrt{\frac{j}{4\pi}}\,,
\end{aligned} \tag{3.257}$$

while at the "South" pole of the unit sphere, one has

$$\begin{aligned}
\hat{\mathrm{e}}_z \cdot \vec{Y}^{j-1\,*}_{j\,\mu}(\pi,\varphi) &= \sqrt{\frac{(j-\mu)\,(j+\mu)}{j(2j-1)}}\,Y_{j-1,\mu}(\pi,\varphi) = (-1)^{j-1}\sqrt{\frac{(j-\mu)\,(j+\mu)}{j(2j-1)}}\,\delta_{\mu 0}\sqrt{\frac{2(j-1)+1}{4\pi}} \\
&= (-1)^{j-1}\,\delta_{\mu 0}\sqrt{\frac{j^2}{j(2j-1)}}\sqrt{\frac{2j-1}{4\pi}} = (-1)^{j-1}\,\delta_{\mu 0}\sqrt{\frac{j}{4\pi}}\,.
\end{aligned} \tag{3.258}$$

Let us also investigate, at the "North" pole of the unit sphere, using Eq. (3.173c),

$$\begin{aligned}
\hat{\mathrm{e}}_z \cdot \vec{Y}^{j+1\,*}_{j\,\mu}(0,\varphi) &= -\sqrt{\frac{(j-\mu+1)\,(j+\mu+1)}{(j+1)(2j+3)}}\,Y_{j+1,\mu}(0,\varphi) \\
&= -\sqrt{\frac{(j-\mu+1)\,(j+\mu+1)}{(j+1)(2j+3)}}\,\delta_{\mu 0}\sqrt{\frac{2(j+1)+1}{4\pi}} \\
&= -\,\delta_{\mu 0}\sqrt{\frac{(j+1)^2}{(j+1)\,(2j+3)}}\sqrt{\frac{2j+3}{4\pi}} = -\delta_{\mu 0}\sqrt{\frac{j+1}{4\pi}}\,.
\end{aligned} \tag{3.259}$$

Conversely, at the "South" pole of the unit sphere, one has

$$\begin{aligned}
\hat{\mathrm{e}}_z \cdot \vec{Y}^{j+1\,*}_{j\,\mu}(\pi,\varphi) &= -\sqrt{\frac{(j-\mu+1)\,(j+\mu+1)}{(j+1)(2j+3)}}\,Y_{j+1,\mu}(\pi,\varphi) \\
&= -\,(-1)^{j+1}\sqrt{\frac{(j-\mu+1)\,(j+\mu+1)}{(j+1)(2j+3)}}\,\delta_{\mu 0}\sqrt{\frac{2(j+1)+1}{4\pi}} \\
&= -\,(-1)^{j+1}\,\delta_{\mu 0}\sqrt{\frac{(j+1)^2}{(j+1)\,(2j+3)}}\sqrt{\frac{2j+3}{4\pi}} = -(-1)^{j-1}\,\delta_{\mu 0}\sqrt{\frac{j+1}{4\pi}}\,.
\end{aligned} \tag{3.260}$$

From Eq. (3.186b), we recall that

$$\vec{N}^{(K)}_{j\mu}(k,\vec{r}) = \sqrt{\frac{j+1}{2j+1}}\, f^{(K)}_{j-1}(k\,r)\, \vec{Y}^{j-1}_{j\,\mu}(\theta,\varphi) - \sqrt{\frac{j}{2j+1}}\, f^{(K)}_{j+1}(k\,r)\, \vec{Y}^{j+1}_{j\,\mu}(\theta,\varphi)\,. \tag{3.261}$$

We may now calculate the electric multipoles,

$$\begin{aligned} n_{j\mu} &= \int \mathrm{d}^3 r\, \vec{J}_0(\vec{r}) \cdot \vec{N}^{(0)*}_{j\mu}(\vec{r}) \\ &= \int \mathrm{d}^3 r \left(j_{j-1}(k\,r) \sqrt{\frac{j+1}{2j+1}}\, \vec{J}_0(\vec{r}) \cdot \vec{Y}^{j-1*}_{j\mu}(\theta,\varphi) - j_{j+1}(k\,r) \sqrt{\frac{j}{2j+1}}\, \vec{J}_0(\vec{r}) \cdot \vec{Y}^{j+1*}_{j\mu}(\theta,\varphi) \right) \\ &= \frac{I_0}{2\pi} \int \mathrm{d}\varphi \int \mathrm{d}r\, r^2\, \cos(kr) \left(j_{j-1}(k\,r) \sqrt{\frac{j+1}{2j+1}} \left[\hat{\mathrm{e}}_z \cdot \vec{Y}^{j-1*}_{j\mu}(0,\varphi) + \hat{\mathrm{e}}_z \cdot \vec{Y}^{j-1*}_{j\mu}(\pi,\varphi) \right] \right. \\ &\qquad \left. - j_{j+1}(k\,r) \sqrt{\frac{j}{2j+1}} \left[\hat{\mathrm{e}}_z \cdot \vec{Y}^{j+1*}_{j\mu}(0,\varphi) + \hat{\mathrm{e}}_z \cdot \vec{Y}^{j+1*}_{j\mu}(\pi,\varphi) \right] \right) . \end{aligned} \tag{3.262}$$

With the help of the results (3.257), (3.258), (3.259) and (3.260), one simplifies this expression to the form

$$\begin{aligned} n_{j\mu} &= \frac{I_0}{2\pi} \int \mathrm{d}\varphi \int \mathrm{d}r\, \cos(kr) \left(j_{j-1}(k\,r) \sqrt{\frac{j+1}{2j+1}} \left[1 + (-1)^{j-1}\right] \delta_{\mu 0} \sqrt{\frac{j}{4\pi}} \right. \\ &\qquad \left. - j_{j+1}(k\,r) \sqrt{\frac{j}{2j+1}} \left(- \left[1 + (-1)^{j-1}\right] \delta_{\mu 0} \sqrt{\frac{j+1}{4\pi}} \right) \right) \\ &= \frac{I_0}{2\pi}\, \delta_{\mu 0} \int \mathrm{d}\varphi \int \mathrm{d}r\, \cos(kr) \sqrt{\frac{j\,(j+1)}{4\pi\,(2j+1)}} \left[1 + (-1)^{j-1}\right] \left(j_{j-1}(k\,r) + j_{j+1}(k\,r) \right) \\ &= I_0\, \delta_{\mu 0} \sqrt{\frac{j\,(j+1)}{4\pi\,(2j+1)}} \left[1 + (-1)^{j-1}\right] \int \mathrm{d}r\, \cos(kr)\, \left(j_{j-1}(k\,r) + j_{j+1}(k\,r) \right) . \end{aligned} \tag{3.263}$$

Recalling the recursion relation for spherical Bessel functions,

$$j_{j-1}(x) + j_{j+1}(x) = \frac{2j+1}{x}\, j_j(x)\,, \tag{3.264}$$

The multipole moment is given as follows,

$$\begin{aligned} n_{j\mu} &= I_0\, \delta_{\mu 0} \sqrt{\frac{j\,(j+1)}{4\pi\,(2j+1)}} \left[1 + (-1)^{j-1}\right] \int \mathrm{d}r\, \cos(kr)\, \frac{2j+1}{k\,r}\, j_j(k\,r) \\ &= -\, I_0\, \delta_{\mu 0} \sqrt{\frac{j\,(j+1)(2j+1)}{4\pi}} \left[1 + (-1)^{j-1}\right] \int \mathrm{d}r \left(- \frac{\cos(kr)}{k\,r} \right) j_j(k\,r) \\ &= -\, I_0\, \delta_{\mu 0} \sqrt{\frac{j\,(j+1)(2j+1)}{4\pi}} \left[1 + (-1)^{j-1}\right] \int\limits_0^{\lambda/4} \mathrm{d}r\, y_0(k\,r)\, j_j(k\,r) \end{aligned} \tag{3.265}$$

where y_0 of course is the spherical Neumann function. The spherical Bessel and Neumann functions of course fulfill the following differential equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{2}{x} \frac{\partial}{\partial x} - \frac{j\,(j+1)}{x^2} - 1 \right) f^{(K)}_j(x) = 0\,, \tag{3.266}$$

where $f_j^{(K)}$ can be j_j or y_j. Hence, one can show that

$$\int \mathrm{d}x\, f_j(x)\, g_{j'}(x) = \frac{x^2}{j'(j'+1) - j(j+1)} \left[f_j(x)\, g'_{j'}(x) - f'_j(x)\, g_{j'}(x)\right] . \tag{3.267}$$

Now we set $f_j = y_0$ and $g_{j'} = j_j$, and have

$$\int \mathrm{d}x\, y_0(x)\, j_j(x) = \frac{x^2}{j(j+1)} \left(y_0(x)\, j'_j(x) - y'_0(x)\, j_j(x)\right) . \tag{3.268}$$

For the case of interest, we have in view of $k = 2\pi/\lambda$, as well as $y_0(\pi/2) = 0$, and $y'_0(\pi/2) = 2/\pi$,

$$\begin{aligned}\int\limits_0^{\lambda/4} \mathrm{d}r\, y_0(k\, r)\, j_j(k\, r) &= \frac{1}{k} \int\limits_0^{\pi/2} \mathrm{d}x\, y_0(x)\, j_j(x) \\ &= -\frac{1}{k}\, \frac{(\pi/2)^2}{j(j+1)}\, y'_0(\pi/2)\, j_j(\pi/2) = -\frac{\pi}{2k} \frac{1}{j(j+1)}\, j_j(\pi/2)\,.\end{aligned} \tag{3.269}$$

Hence,

$$n_{j\mu} = \frac{\pi\, I_0}{2k}\, \delta_{\mu 0} \sqrt{\frac{2j+1}{4\pi\, j\,(j+1)}} \left[1 + (-1)^{j-1}\right] j_j(\pi/2)\,. \tag{3.270}$$

The results (3.255) and (3.270) enter the relations (3.197c) and (3.197d),

$$\vec{E}_0(\vec{r}) = -k^2\, c\, \mu_0 \sum_{j=0}^{\infty} \sum_{\mu=-j}^{j} \left(m_{j\mu}\, \vec{M}^{(1)}_{j\mu}(k, \vec{r}) + n_{j\mu}\, \vec{N}^{(1)}_{j\mu}(k, \vec{r})\right) = -k^2\, c\, \mu_0 \sum_{j=0}^{\infty} n_{j0}\, \vec{N}^{(1)}_{j0}(k, \vec{r})\,, \tag{3.271}$$

and

$$\vec{B}_0(\vec{r}) = k^2\, \mu_0 \sum_{j=0}^{\infty} \sum_{\mu=-j}^{j} \left(m_{j\mu}\, \vec{N}^{(1)}_{j\mu}(k, \vec{r}) - n_{j\mu}\, \vec{M}^{(1)}_{j\mu}(k, \vec{r})\right) = -k^2\, \mu_0 \sum_{j=0}^{\infty} n_{j0}\, \vec{M}^{(1)}_{j0}(k, \vec{r})\,. \tag{3.272}$$

In both cases, for the half-wave antenna, one can drop the $m_{j\mu}$ terms and restrict the sum over the $n_{j\mu}$ terms to those with $\mu = 0$.

We now intend to use Eq. (3.243) in order to evaluate the radiated power in the far field. It is instructive to rewrite Eq. (3.243) somwhat and to introduce the factor

$$\mu_0\, c = \frac{\mu_0}{\sqrt{\epsilon_0\, \mu_0}} = Z_0 \approx 377\, \Omega\,, \tag{3.273}$$

which is commonly referred to as the vacuum impedance. The radiated power P can be written as a product of the vacuum impedance and the square of the current I_0, times some numerical factor,

$$\begin{aligned}P &= \frac{k^2}{2}\, Z_0 \sum_{j\mu} \left\{|m_{j\mu}|^2 + |n_{j\mu}|^2\right\} = \frac{k^2}{2}\, Z_0 \sum_{j\mu} |n_{j\mu}|^2 \\ &= \frac{k^2}{2}\, Z_0 \sum_{j \text{ odd}} |n_{j\,0}|^2 = Z_0\, I_0^2\, \frac{\pi}{8} \sum_{j \text{ odd}} \frac{2j+1}{j\,(j+1)} \left[j_j(\pi/2)\right]^2 .\end{aligned} \tag{3.274}$$

The sum

$$\mathcal{S} = \sum_{j \text{ odd}} \frac{2j+1}{j\,(j+1)} \left[j_j(\pi/2)\right]^2 = 0.246\,986 \tag{3.275}$$

is rapidly converging as j is increased. The term with $j = 1$ is $0.246\,384$, implying that dipole radiation accounts for $99.76\,\%$ of the radiated power. Equating the radiated power with the "resistance to radiation emission", one has

$$P = \frac{1}{2}\, R_{\rm rad}\, I_0^2\,, \qquad R_{\rm rad} = Z_0\, \mathcal{S}\, \frac{\pi}{4} = 73.079\,\Omega\,. \tag{3.276}$$

The physical interpretation is that the antenna acts just like a resistor, as it emits energy in the form of outgoing radiation. The power absorbed by the antenna is proportional to $Z_0\, I_0^2$. For given ϵ_0, if the speed of light were infinitely fast, then in view of $\mu_0\, \epsilon_0 = 1/c^2$, we would have $\mu_0 \to 0$ and $Z_0 \to 0$; it would be "easier" to emit radiation, or in other words, the emitted electromagnetic waves would carry less energy. With a grain of salt, we can remark that, because the impedance of the vacuum has an appreciable numerical value of $\approx 377\,\Omega$, radio communication works well.

3.5.6 Long–Wavelength Limit of the Dipole Term

The somewhat elegant formalism we have discussed, obscures the leading contributions to the dipole terms, and to other, higher-order multipole terms; we shall now attempt to recover these terms, at least in the long-wavelength limit. In particular, we attempt to show that the dipole term is part of $N^(1)_{1\mu}$, and is hidden in the contribution of $j_{j-1}(kr) = j_0(kr)$. It remains to verify the prefactor. Let us therefore investigate

$$n_{j\mu} = \int \mathrm{d}^3 r\, \vec{J}_0(\vec{r}) \cdot \vec{N}^{(0)*}_{j\mu}(k, \vec{r}) \tag{3.277}$$

Now, in view of Eqs. (3.186a) and (3.186b), we have

$$\begin{aligned} \vec{N}^{(0)}_{j\mu}(k,\vec{r}) &= -\frac{\mathrm{i}}{k}\, \vec{\nabla} \times \vec{M}^{(K)}_{j\mu}(k,\vec{r}) \\ &= -\frac{1}{\sqrt{j(j+1)}}\, \frac{1}{k}\, \vec{\nabla} \times \left(\vec{r} \times \vec{\nabla}\right) j_j(k\,r)\, Y^*_{j\mu}(\theta,\varphi)\,. \end{aligned} \tag{3.278}$$

The expression $\vec{\nabla} \times \left(\vec{r} \times \vec{\nabla}\right) j_j(k\,r)\, Y_{j\mu}(\theta,\varphi)$ is tricky; one can simplify it but must remember that the leftmost gradient operator acts on everything that follows. The result is that

$$\vec{\nabla} \times \left(\vec{r} \times \vec{\nabla}\right) f(\vec{r}) = \left[\vec{r}\, \vec{\nabla}^2 - \vec{\nabla}\, \frac{\partial}{\partial r}\, r\right] f(\vec{r})\,. \tag{3.279}$$

This identity deserves some comments. The normal "BAC–CAB" rule states that

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\, (\vec{A} \cdot \vec{C}) - \vec{C}\, (\vec{A} \cdot \vec{B})\,. \tag{3.280}$$

However, here we have to take into account the ordering of the operators:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \sum_i A^i (\vec{B}) C^i - \sum_i A^i\, B^i\, (\vec{C})\,. \tag{3.281}$$

Let us try and be careful,

$$\begin{aligned}
\vec{\nabla}\times(\vec{r}\times\vec{\nabla}) =& \sum_{ij}\frac{\partial}{\partial x^i}\,\hat{\mathrm{e}}_j\,x^j\,\frac{\partial}{\partial x^i}-\sum_{ij}\frac{\partial}{\partial x^i}\,x^i\,\hat{\mathrm{e}}_j\,\frac{\partial}{\partial x^j}\\
=& \sum_{ij}\hat{\mathrm{e}}_j\,\delta^{ij}\,\frac{\partial}{\partial x^i}+\sum_{ij}\hat{\mathrm{e}}_j\,x^j\,\frac{\partial}{\partial x^i}\,\frac{\partial}{\partial x^i}--\sum_{ij}\hat{\mathrm{e}}_j\,\frac{\partial}{\partial x^i}\,\left[x^i,\,\frac{\partial}{\partial x^j}\right]-\sum_{ij}\hat{\mathrm{e}}_j\,\frac{\partial}{\partial x^i}\,\frac{\partial}{\partial x^j}x^i\\
=& \sum_{i}\hat{\mathrm{e}}_i\frac{\partial}{\partial x^i}+\vec{r}\sum_{i}\frac{\partial}{\partial x^i}\,\frac{\partial}{\partial x^i}-\sum_{i}\hat{\mathrm{e}}_j\,\frac{\partial}{\partial x^i}\,(-\delta^{ij})-\sum_{i}\hat{\mathrm{e}}_j\,\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\,x^i\\
=& \vec{\nabla}+\vec{r}\vec{\nabla}^2+\vec{\nabla}-\vec{\nabla}\left(3+r\frac{\partial}{\partial r}\right)=-\vec{\nabla}+\vec{r}\vec{\nabla}^2-\vec{\nabla}\left(r\,\frac{\partial}{\partial r}\right)\\
=& -\vec{\nabla}+\vec{r}\vec{\nabla}^2-\vec{\nabla}\left(\left[r,\frac{\partial}{\partial r}\right]+\frac{\partial}{\partial r}\,r\right)=\vec{r}\vec{\nabla}^2-\vec{\nabla}\left(\frac{\partial}{\partial r}\,r\right).
\end{aligned}\tag{3.282}$$

We have used the relation

$$\sum_i\frac{\partial}{\partial x^i}\,x^i=\sum_i\left[\frac{\partial}{\partial x^i},x^i\right]+\sum_i x^i\,\frac{\partial}{\partial x^i}=3+r\,\frac{\partial}{\partial r}.\tag{3.283}$$

In view of Eq. (3.279), one has

$$\vec{N}^{(0)}_{j\mu}(k,\vec{r})=-\frac{1}{\sqrt{j(j+1)}}\,\frac{1}{k}\,\left[\vec{r}\,\vec{\nabla}^2-\vec{\nabla}\left(\frac{\partial}{\partial r}r\right)\right]j_j(k\,r)\,Y^*_{j\mu}(\theta,\varphi).\tag{3.284}$$

One uses Eq. (3.279), the fact that $j_j(k\,r)\,Y_{j\mu}(\theta,\varphi)$ satisfies the Helmholtz equation, integration by parts, and the continuity equation, to write

$$\begin{aligned}
n_{j\mu} =& -\frac{1}{\sqrt{j(j+1)}}\,\frac{1}{k}\,\int\mathrm{d}^3r\,\vec{J}_0(\vec{r})\cdot\left[\vec{r}\,\vec{\nabla}^2-\vec{\nabla}\left(\frac{\partial}{\partial r}r\right)\right]j_j(k\,r)\,Y^*_{j\mu}(\theta,\varphi)\\
=& -\frac{1}{\sqrt{j(j+1)}}\,\frac{1}{k}\,\int\mathrm{d}^3r\,\vec{J}_0(\vec{r})\cdot\left[-\vec{r}\,k^2-\vec{\nabla}\left(\frac{\partial}{\partial r}r\right)\right]j_j(k\,r)\,Y^*_{j\mu}(\theta,\varphi)\\
=& \frac{1}{\sqrt{j(j+1)}}\,\frac{1}{k}\,\int\mathrm{d}^3r\,\left[k^2\,j_j(k\,r)\vec{r}\cdot\vec{J}_0(\vec{r})-\left(\frac{\partial}{\partial r}rj_j(k\,r)\right)\vec{\nabla}\cdot\vec{J}_0(\vec{r})\right]\,Y^*_{j\mu}(\theta,\varphi)\\
=& \frac{1}{\sqrt{j(j+1)}}\,\int\mathrm{d}^3r\,\left[k\,j_j(k\,r)\,\vec{r}\cdot\vec{J}_0(\vec{r})-\left(\frac{\partial}{\partial r}rj_j(k\,r)\right)\,(\mathrm{i}c\,\rho_0(\vec{r}))\right]Y^*_{j\mu}(\theta,\varphi).
\end{aligned}\tag{3.285}$$

In the long-wavelength limit, i.e, small-k limit, the dominant term obviously is the second one, as it carries no explicit factor k. Using the long-wavelength (small-k) asymptotics of the Bessel function, we have

$$\begin{aligned}
n_{j\mu} =& -\frac{\mathrm{i}c}{\sqrt{j(j+1)}}\,\int\mathrm{d}^3r\,\left(\frac{\partial}{\partial r}rj_j(k\,r)\right)\,Y^*_{j\mu}(\theta,\varphi)\,\rho_0(\vec{r})\\
\approx& -\frac{\mathrm{i}c}{\sqrt{j(j+1)}}\,\int\mathrm{d}^3r\,\left(\frac{\partial}{\partial r}\frac{k^j\,r^{j+1}}{(2j+1)!!}\right)\,Y^*_{j\mu}(\theta,\varphi)\,\rho_0(\vec{r})\\
\approx& -\frac{\mathrm{i}\,k^j\,c}{\sqrt{j(j+1)}}\,(j+1)\,\int\mathrm{d}^3r\,\left(\frac{r^j}{(2j+1)!!}\right)\,Y^*_{j\mu}(\theta,\varphi)\,\rho_0(\vec{r})\\
\approx& -\mathrm{i}\,k^j\,c\,\sqrt{\frac{j+1}{j}}\,\frac{1}{(2j+1)!!}\,\int\mathrm{d}^3r\,r^j\,Y^*_{j\mu}(\theta,\varphi)\,\rho_0(\vec{r})\\
\approx& -\frac{\mathrm{i}\,c}{(2j+1)!!}\,\sqrt{\frac{j+1}{j}}\,k^j\,q^{(0)}_{j\mu}.
\end{aligned}\tag{3.286}$$

Here, the $q^{(0)}_{j\mu}$ are formally equal to the multipole moments, as familiar from electrostatics, but this time evaluated using the spatial part $\rho_0(\vec{r})$ of the oscillating charge density $\rho(\vec{r},t) = \rho_0(\vec{r}) \exp(-\mathrm{i}\omega t)$. Setting $j = 1$, one recovers the long-wavelength limit of the electric-dipole term.

3.6 Potentials due to Moving Charges

Up to now, we have used our formalism in order to describe harmonic oscillatory fields. However, our formulas actually are more general, and as an example for an alternative application, we consider the potentials generated by a moving charge. These are otherwise called the Liénard–Wiechert potentials.

The charge and current densities are

$$\boxed{\text{Charge Density of Moving Charge:}} \qquad \rho\left(\vec{r},t\right) = q\,\delta^{(3)}\left(\vec{r}-\vec{R}\left(t\right)\right), \tag{3.287a}$$

$$\boxed{\text{Current Density of Moving Charge:}} \qquad \vec{J}\left(\vec{r},t\right) = q\,\delta^{(3)}\left(\vec{r}-\vec{R}\left(t\right)\right)\left[\frac{\mathrm{d}}{\mathrm{d}t}\vec{R}\left(t\right)\right]. \tag{3.287b}$$

If we ignore the homogeneous terms in Eqs. (3.7a) and (3.7b) and keep the integration over d^3r' and $\mathrm{d}t'$ for the time being, then

$$\Phi\left(\vec{r},t\right) = \int \mathrm{d}^3r'\mathrm{d}t'\left[\rho\left(\vec{r}',t'\right)\right]\left\{\Theta\left(t-t'\right)\frac{1}{4\pi\epsilon_0|\vec{r}-\vec{r}'|}\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)\right\}, \tag{3.288a}$$

$$\vec{A}\left(\vec{r},t\right) = \int \mathrm{d}^3r'\mathrm{d}t'\left[\frac{1}{c^2}\vec{J}\left(\vec{r}',t'\right)\right]\left\{\Theta\left(t-t'\right)\frac{1}{4\pi\epsilon_0|\vec{r}-\vec{r}'|}\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)\right\}. \tag{3.288b}$$

Using the explicit expressions for the potentials of the moving charges in Eq. (3.287), we obtain

$$\Phi\left(\vec{r},t\right) = \frac{q}{4\pi\epsilon_0}\int \mathrm{d}^3r'\mathrm{d}t'\left[\delta^{(3)}\left(\vec{r}'-\vec{R}\left(t'\right)\right)\right]\left\{\frac{\Theta\left(t-t'\right)}{|\vec{r}-\vec{r}'|}\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)\right\}, \tag{3.289a}$$

$$\vec{A}\left(\vec{r},t\right) = \frac{q}{4\pi\epsilon_0 c^2}\int \mathrm{d}^3r'\mathrm{d}t'\left[\delta^{(3)}\left(\vec{r}'-\vec{R}\left(t'\right)\right)\dot{\vec{R}}\left(t'\right)\right]\left\{\frac{\Theta\left(t-t'\right)}{|\vec{r}-\vec{r}'|}\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)\right\}. \tag{3.289b}$$

For $c \to \infty$, these expressions become trivial,

$$\Phi\left(\vec{r},t\right) \stackrel{c\to\infty}{=} \frac{q}{4\pi\epsilon_0}\,\frac{1}{|\vec{r}-\vec{R}(t)|}, \tag{3.290a}$$

$$\vec{A}\left(\vec{r},t\right) \stackrel{c\to\infty}{=} \frac{1}{c}\,\frac{q}{4\pi\epsilon_0}\,\frac{\dot{\vec{R}}(t)/c}{|\vec{r}-\vec{R}(t)|}, \tag{3.290b}$$

which are the results obtained without retardation.

Contrary to our approach for the radiating source, we do not carry out the time integration over $\mathrm{d}t'$ but the integration over d^3r', as this eliminates three, not one, integrations at once,

$$\Phi\left(\vec{r},t\right) = \frac{q}{4\pi\epsilon_0}\int \mathrm{d}t'\,\Theta\left(t-t'\right)\,\frac{1}{|\vec{r}-\vec{R}(t')|}\,\delta\left(t'-\left(t-\frac{|\vec{r}-\vec{R}(t')|}{c}\right)\right), \tag{3.291a}$$

$$\vec{A}\left(\vec{r},t\right) = \frac{q}{4\pi\epsilon_0 c^2}\int \mathrm{d}t'\,\Theta\left(t-t'\right)\,\frac{\dot{\vec{R}}(t')}{|\vec{r}-\vec{R}(t')|}\,\delta\left(t'-\left(t-\frac{|\vec{r}-\vec{R}(t')|}{c}\right)\right). \tag{3.291b}$$

The δ function peaks at

$$t' = t_{\rm ret} = t - \frac{|\vec{r} - \vec{R}(t')|}{c} = t - \frac{|\vec{r} - \vec{R}(t_{\rm ret})|}{c} < t\,, \tag{3.292}$$

or

$$t_{\rm ret} = t - \frac{|\vec{r} - \vec{R}(t_{\rm ret})|}{c}\,, \qquad\qquad c(t - t_{\rm ret})^2 = \left(\vec{r} - \vec{R}(t_{\rm ret})\right)^2\,, \tag{3.293}$$

so that the step function is always unity at the point where the Dirac-δ peaks. That means that all points on the trajectory of the particle for which the retardation condition is fulfilled, contribute to the integrals. Let us assume that the Dirac-δ peaks only once, namely, at $t' = t_{\rm ret}$. The integration over $\mathrm{d}t'$ in Eq. (3.291) still leads to a nontrivial Jacobian as the argument of the Dirac-δ is nontrivial and needs to be differentiated with respect to t'. Indeed, we have, using the three-dimensional chain rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t'}\left(t' - t + \frac{|\vec{r} - \vec{R}(t')|}{c}\right) &= 1 + \frac{1}{c}\,\frac{\vec{r} - \vec{R}(t')}{|\vec{r} - \vec{R}(t')|}\cdot\frac{\mathrm{d}}{\mathrm{d}t'}\left(-\vec{R}(t')\right) \\ &= 1 - \frac{1}{c}\,\frac{\mathrm{d}\vec{R}(t')}{\mathrm{d}t'}\cdot\frac{\vec{r} - \vec{R}(t')}{|\vec{r} - \vec{R}(t')|}\,, \qquad\qquad t' = t_{\rm ret}\,.\end{aligned} \tag{3.294}$$

So, we have

$$\Phi\left(\vec{r}, t\right) = \frac{q}{4\pi\epsilon_0}\,\frac{1}{|\vec{r} - \vec{R}(t_{\rm ret})|}\,\left(1 - \frac{\mathrm{d}}{\mathrm{d}t'}\,\frac{|\vec{r} - \vec{R}(t')|}{c}\bigg|_{t'=t_{\rm ret}}\right)^{-1}\,, \tag{3.295a}$$

$$\vec{A}\left(\vec{r}, t\right) = \frac{1}{c}\,\frac{q}{4\pi\epsilon_0}\,\frac{\dot{\vec{R}}(t_{\rm ret})/c}{|\vec{r} - \vec{R}(t_{\rm ret})|}\,\left(1 - \frac{\mathrm{d}}{\mathrm{d}t'}\,\frac{|\vec{r} - \vec{R}(t')|}{c}\bigg|_{t'=t_{\rm ret}}\right)^{-1}\,, \tag{3.295b}$$

which can be written as

$$\boxed{\textbf{Scalar Potential:}} \qquad \Phi\left(\vec{r}, t\right) = \frac{q}{4\pi\epsilon_0}\,\frac{1}{|\vec{r} - \vec{R}(t_{\rm ret})|}\,\left(1 - \frac{\dot{\vec{R}}(t_{\rm ret})}{c}\cdot\frac{\vec{r} - \vec{R}(t_{\rm ret})}{|\vec{r} - \vec{R}(t_{\rm ret})|}\right)^{-1}\,, \tag{3.296a}$$

$$\boxed{\textbf{Vector Potential:}} \qquad \vec{A}\left(\vec{r}, t\right) = \frac{q}{4\pi\epsilon_0 c^2}\,\frac{\dot{\vec{R}}(t_{\rm ret})}{|\vec{r} - \vec{R}(t_{\rm ret})|}\,\left(1 - \frac{\dot{\vec{R}}(t_{\rm ret})}{c}\cdot\frac{\vec{r} - \vec{R}(t_{\rm ret})}{|\vec{r} - \vec{R}(t_{\rm ret})|}\right)^{-1}\,, \tag{3.296b}$$

and constitute the Liénard–Wiechert potentials. With the definitions

$$\vec{\beta} \equiv \vec{\beta}(t_{\rm ret}) = \frac{\dot{\vec{R}}(t_{\rm ret})}{c}\,, \quad \hat{n} \equiv \frac{\vec{r} - \vec{R}(t_{\rm ret})}{|\vec{r} - \vec{R}(t_{\rm ret})|}\,, \quad \mathcal{R} \equiv |\vec{r} - \vec{R}(t_{\rm ret})|\,, \tag{3.297}$$

we can write the scalar and vector potentials in the familiar form,

$$\Phi\left(\vec{r}, t\right) = \frac{q}{4\pi\epsilon_0}\,\frac{1}{\mathcal{R}\left(1 - \vec{\beta}\cdot\hat{n}\right)}\,, \tag{3.298a}$$

$$\vec{A}\left(\vec{r}, t\right) = \frac{q}{4\pi\epsilon_0 c}\,\frac{\vec{\beta}}{\mathcal{R}\left(1 - \vec{\beta}\cdot\hat{n}\right)}\,. \tag{3.298b}$$

In Gaussian units, these expressions are given in Eq. (14.8) in Chapter 14 of [J. D. Jackson, *Classical Electrodynamics*, (John Wiley and Sons, New York, 1998)]. The four-vector is given by the combination $\mathcal{A}^\mu = (\Phi, c\,\vec{A})$.

Let us assume that we are given $\vec{r}$ and t. It is highly nontrivial to find the retarded time $t_{\rm ret}$ and the point $\vec{R}(t_{\rm ret})$ along the particle trajectory that fulfill Eq. (3.293), namely, $c(t - t_{\rm ret})^2 = (\vec{r} - \vec{R}(t_{\rm ret}))^2$, even if we have a uniform motion. The reason is that the vector $\vec{r} - \vec{R}(t_{\rm ret})$ can have any direction in space. One possible strategy is to define

$$W = c\,(t - t_{\rm ret}) \tag{3.299}$$

and to express $\vec{r} - \vec{R}(t_{\rm ret})$ in terms of W,

$$\vec{r} - \vec{R}(t_{\rm ret}) = \vec{r} - \vec{R}(t - W/c)\,. \tag{3.300}$$

The condition then is

$$\left(\vec{r} - \vec{R}(t - W/c)\right)^2 = W^2\,, \tag{3.301}$$

which is a quadratic equation for W. For uniform motion,

$$\vec{R}\,(t) = \vec{v}_0\;t\,, \tag{3.302}$$

So, for uniform motion, Eq. (3.301) becomes

$$\left(\vec{r} - \vec{v}_0\,t + \frac{\vec{v}_0}{c}\;W\right)^2 = W^2\,. \tag{3.303}$$

With the definition

$$\vec{R}_0 = \vec{R}_0(t) \equiv \vec{r} - \vec{v}_0\,t\,, \qquad \vec{\beta}_0 = \frac{\vec{v}_0}{c}\,, \tag{3.304}$$

we can formulate the quadratic equation,

$$\left(\vec{R}_0 + \frac{\vec{v}_0}{c}\;W\right)^2 = W^2 = \left(\vec{R}_0 + \vec{\beta}_0\;W\right)^2 = \vec{R}_0^2 + 2\vec{R}_0\cdot\vec{\beta}_0\;W + \beta_0^2\,W^2\,. \tag{3.305}$$

Completing the square,

$$\left(1 - \beta_0^2\right)\;W^2 - 2\vec{R}_0\cdot\vec{\beta}_0\;W - \vec{R}_0^2 = 0 = \left(\sqrt{1-\beta_0^2}\,W - \frac{\vec{R}_0\cdot\vec{\beta}_0}{\sqrt{1-\beta_0^2}}\right)^2 - \vec{R}_0^2 - \frac{(\vec{R}_0\cdot\vec{\beta}_0)^2}{1-\beta_0^2}\,. \tag{3.306}$$

Now, with the definition of the Lorentz factor,

$$\gamma_0 = \frac{1}{\sqrt{1-\beta_0^2}}\,, \tag{3.307}$$

we can finally solve for W and, thus, for $t - t_{\rm ret}$,

$$\begin{aligned}
\Rightarrow \qquad & \left(\frac{W}{\gamma_0} - \gamma_0\,\vec{R}_0\cdot\vec{\beta}_0\right)^2 = \vec{R}_0^2 + \gamma_0^2\,(\vec{R}_0\cdot\vec{\beta}_0)^2 \\
\Rightarrow \qquad & c\,(t - t_{\rm ret}) = W = \gamma_0^2\,\vec{R}_0\cdot\vec{\beta}_0 \pm \gamma_0\,\sqrt{\gamma_0^2\,(\vec{R}_0\cdot\vec{\beta}_0)^2 + \vec{R}_0^2} \\
\Rightarrow \qquad & (t - t_{\rm ret}) = \frac{\gamma_0^2}{c}\,\vec{R}_0\cdot\vec{\beta}_0\left(1 \pm \sqrt{1 + \frac{\vec{R}_0^2}{\gamma_0^2\;(\vec{R}_0\cdot\vec{\beta}_0)^2}}\right) \\
\Rightarrow \qquad & (t - t_{\rm ret}) = \frac{\gamma_0^2}{c}\,\vec{R}_0\cdot\vec{\beta}_0\left(1 + \sqrt{1 + \frac{\vec{R}_0^2}{\gamma_0^2\;(\vec{R}_0\cdot\vec{\beta}_0)^2}}\right) > 0\,.
\end{aligned} \tag{3.308}$$

The solution for the retarded time as a function of t then is

$$t_{\rm ret} = t - \frac{\gamma_0^2}{c}\,\vec{R}_0\cdot\vec{\beta}_0\left(1 + \sqrt{1 + \frac{\vec{R}_0^2}{\gamma_0^2\;(\vec{R}_0\cdot\vec{\beta}_0)^2}}\right) < t\,. \tag{3.309}$$

where
$$\vec{\beta}_0 = \frac{\vec{v}_0}{c}\,, \qquad \beta_0 = \frac{|\vec{v}_0|}{c}\,, \qquad \gamma_0 = \frac{1}{\sqrt{1-\beta_0^2}} \qquad \vec{R}_0 = \vec{r} - \vec{v}_0 t\,. \tag{3.310}$$

We now investigate the scalar potential (3.296a) in the limit where $\vec{R}_0$ and $\vec{\beta}_0$ are almost parallel, setting
$$\frac{\vec{R}_0 \cdot \vec{\beta}_0}{|\vec{R}_0|\,\beta_0} = \cos(\theta_0) \equiv 1 - \frac{1}{2!}\,\theta_0^2 + \mathcal{O}(\theta_0^4)\,. \tag{3.311}$$
and perform an expansion in powers of θ_0, about the point where the potential is evaluated almost on the linear trajectory of the uniformly moving particle. In doing so, we calculate the structure of the relativistic corrections, which are manifest in the higher-order terms in β_0. It is advantageous to set
$$\vec{r} - \vec{R}(t_{\rm ret}) = \vec{R}_0 + \vec{\beta}_0\, W\,, \tag{3.312}$$
which implies that with $R_0 = |\vec{R}_0|$, $\beta_0 = |\vec{\beta}_0|$,
$$|\vec{r} - \vec{R}(t_{\rm ret})| = |\vec{R}_0 + \vec{\beta}_0\, W| = \sqrt{R_0^2 + 2\beta_0 R_0 \cos\theta_0\, W + \beta_0^2\, W^2}\,. \tag{3.313}$$
This implies that
$$\begin{aligned}
\frac{1}{|\vec{r} - \vec{R}(t_{\rm ret})|}\left(1 - \frac{\dot{\vec{R}}(t_{\rm ret})}{c}\cdot\frac{\vec{r} - \vec{R}(t_{\rm ret})}{|\vec{r} - \vec{R}(t_{\rm ret})|}\right)^{-1} &= \frac{1}{|\vec{R}_0 + \vec{\beta}_0\, W|}\left(1 - \vec{\beta}_0 \cdot \frac{\vec{R}_0 + \vec{\beta}_0\, W}{|\vec{R}_0 + \vec{\beta}_0\, W|}\right)^{-1} \\
&= \left(|\vec{R}_0 + \vec{\beta}_0\, W| - \vec{\beta}_0 \cdot \left(\vec{R}_0 + \vec{\beta}_0\, W\right)\right)^{-1} \\
&= \left(\sqrt{R_0^2 + 2\,\beta_0\, R_0\, \cos\theta_0\, W + \beta_0^2\, W^2} - \beta_0\, R_0\, \cos\theta_0 - \beta_0^2\, W\right)^{-1}\,.
\end{aligned} \tag{3.314}$$
The scalar potential then is
$$\begin{aligned}
\Phi\,(\vec{r}, t) &= \frac{q}{4\pi\epsilon_0}\,\frac{1}{|\vec{r} - \vec{R}(t_{\rm ret})|}\left(1 - \frac{\dot{\vec{R}}(t_{\rm ret})}{c}\cdot\frac{\vec{r} - \vec{R}(t_{\rm ret})}{|\vec{r} - \vec{R}(t_{\rm ret})|}\right)^{-1} \\
&= \frac{q}{4\pi\epsilon_0\, R_0}\,\frac{R_0}{\sqrt{R_0^2 + 2\,\beta_0\, R_0\, \cos\theta_0\, W + \beta_0^2\, W^2} - \beta_0\, R_0\, \cos\theta_0 - \beta_0^2\, W}\,.
\end{aligned} \tag{3.315}$$
Using the result
$$W = \beta_0\, \gamma_0^2\, R_0\, \cos\theta_0\, \left(1 + \sqrt{1 + \frac{1}{\gamma_0^2\, \beta_0\, \cos^2\theta_0}}\right)\,, \tag{3.316}$$
and the Taylor expansion
$$\cos\theta_0 = 1 - \frac{\theta_0^2}{2!} + \mathcal{O}(\theta_0^4)\,, \tag{3.317}$$
one can finally derive after expanding in powers of θ_0,
$$\Phi\,(\vec{r}, t) = \frac{q}{4\pi\epsilon_0}\,\frac{1}{|\vec{r} - \vec{v}_0\, t|}\left(1 - \frac{\beta_0^2\,\theta_0^2}{2} + \mathcal{O}(\theta_0^4)\right)^{-1} = \frac{q}{4\pi\epsilon_0}\,\frac{1}{|\vec{r} - \vec{v}_0\, t|}\left(1 + \frac{\beta_0^2\,\theta_0^2}{2} + \mathcal{O}(\theta_0^4)\right)\,. \tag{3.318}$$
The identity
$$\vec{r} - \vec{v}_0\, t + \frac{\vec{v}_0}{c}\, c\,(t - t_{\rm ret}) = r - \vec{v}_0\, t_{\rm ret} = \vec{r} - \vec{R}(t_{\rm ret}) \tag{3.319}$$
comes in handy.

Chapter 4

Electromagnetic Waves in Waveguides and Cavities

4.1 Orientation

In vacuum, the modes of the electromagnetic field are not restricted. Waves can travel everywhere, and a typical space-time dependence is of the form

$$\exp(\mathrm{i}\,\vec{k}\cdot\vec{r}-\mathrm{i}\,\omega t)\,. \tag{4.1}$$

The boundary conditions induced by waveguides and cavities limit the admissible values of $\vec{k}$ and ω to specific, countable values. In some sense, the boundary conditions induce a quantization of the modes available to the electromagnetic field,

$$\vec{k}=\vec{k}_n\,,\qquad \omega=\omega_n\,,\qquad \text{where } n \text{ is a multi-index describing the modes.} \tag{4.2}$$

In some sense, the waves are "bound" into the cavities, and the additional quantization conditions induced by the cavity correspond to the "bound states" in atoms, which are also discrete (yet, the atomic bound states are infinitely many, just countable).

In the following sections, we discuss the modes available to electromagnetic wave propagation in waveguides, and also, the energy spectrum for electromagnetic modes bound in or "into" a cavity.

As an additional application, we will discuss the Casimir effect between plates, which is equivalent to the attraction of two perfectly conducting plates due to vacuum fluctuations of the light field.

4.2 Waveguides

4.2.1 General Formalism

Our aim in this section is to see if we can find additional, important relations for waves traveling in a waveguide, e.g., if we can express the E_x and E_y components as functions of E_z only (for the electric field), and likewise for the magnetic field. We shall thus consider an electromagnetic field propagating inside a hollow (in the present case cylindrical or rectangular) conductor/wave guide. There are no sources inside the conductor, but we shall assume the material inside the wave guide to be isotropic with electric permittivity

$$\tilde{\epsilon}(\omega) = \epsilon_r(\omega)\epsilon_0 \,, \qquad \epsilon_r \equiv \epsilon_r(\omega) \,, \tag{4.3}$$

and magnetic permeability,

$$\tilde{\mu}(\omega) = \tilde{\mu}_r(\omega)\, \mu_0 \,, \qquad \mu_r \equiv \tilde{\mu}_r(\omega) \,. \tag{4.4}$$

Without boundary conditions, the speed of the propagating wave in the medium is $c/\sqrt{\epsilon_r \mu_r}$. It is more economical to leave out the tilde over the symbols ϵ and μ in the notation, and we shall denote the fields and their Fourier transforms by the same symbols in the following. The direction of propagation is along the cylindrical axis, which is the positive $\hat{e}_z$ direction. We will work in mixed position-frequency space, with fields $\vec{E} = \vec{E}(\vec{r}, \omega)$ and $\vec{B} = \vec{B}(\vec{r}, \omega)$, suppressing the explicit notation for the Fourier representation using a tilde. Later on, we will specialize to a single mode with $\vec{E}(\vec{r}, t) = \vec{E}(\vec{r}) \exp(-\mathrm{i}\omega t)$ and $\vec{B}(\vec{r}, t) = \vec{B}(\vec{r}) \exp(-\mathrm{i}\omega t)$. The Maxwell equations yield, with $\epsilon = \epsilon_r\, \epsilon_0$ and $\mu = \mu_r \mu_0$, in a source-free region,

$$\vec{\nabla} \times \vec{E}(\vec{r}, t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r}, t) \qquad \Rightarrow \qquad \vec{\nabla} \times \vec{E}(\vec{r}, \omega) = \mathrm{i}\omega \vec{B}(\vec{r}, \omega) \,, \tag{4.5a}$$

$$\vec{\nabla} \times \vec{H}(\vec{r}, t) = \frac{\partial}{\partial t}\vec{D}(\vec{r}, t) \qquad \Rightarrow \qquad \vec{\nabla} \times \vec{B}(\vec{r}, \omega) = -\mathrm{i}\,\frac{\omega}{c^2}\,\epsilon_r \mu_r \vec{E}(\vec{r}, \omega) \,. \tag{4.5b}$$

Inserting the curl of the first equation into the second, or vice versa, one obtains the following wave equations,

$$\left(\vec{\nabla}^2 + \epsilon_r\, \mu_r\, \frac{\omega^2}{c^2}\right)\, \vec{B}(\vec{r}, \omega) = \vec{0} \,, \tag{4.6}$$

$$\left(\vec{\nabla}^2 + \epsilon_r\, \mu_r\, \frac{\omega^2}{c^2}\right)\, \vec{E}(\vec{r}, \omega) = \vec{0} \,. \tag{4.7}$$

From now on, we work in frequency space and restrict the discussion to a single Fourier mode,

$$\vec{E}(\vec{r}, t) = \vec{E}(\vec{r})\, \exp(-\mathrm{i}\omega t) \,, \qquad \vec{E}(\vec{r}, \omega') = \vec{E}(\vec{r})\, 2\pi\, \delta(\omega' - \omega) \,, \tag{4.8}$$

$$\vec{B}(\vec{r}, t) = \vec{B}(\vec{r})\, \exp(-\mathrm{i}\omega t) \,, \qquad \vec{B}(\vec{r}, \omega') = \vec{B}(\vec{r})\, 2\pi\, \delta(\omega' - \omega) \,. \tag{4.9}$$

As the wave is propagating along the positive $\hat{e}_z$ direction, we shall further assume that:

$$\vec{E}(\vec{r}) = \vec{E}(x, y)\, \mathrm{e}^{\mathrm{i}kz} \,, \qquad \vec{B}(\vec{r}) = \vec{B}(x, y)\, \mathrm{e}^{\mathrm{i}kz} \,. \tag{4.10}$$

Thus, the wave equations become

$$\left(\vec{\nabla}_{\parallel}^2 + \epsilon_r\, \mu_r\, \frac{\omega^2}{c^2} - k^2\right)\, \vec{E}(x, y) = \vec{0} \,, \tag{4.11}$$

$$\left(\vec{\nabla}_{\parallel}^2 + \epsilon_r\, \mu_r\, \frac{\omega^2}{c^2} - k^2\right)\, \vec{B}(x, y) = \vec{0} \,. \tag{4.12}$$

where

$$\vec{\nabla}_{\parallel}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \,, \qquad \vec{\nabla}_{\parallel} = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} \,. \tag{4.13}$$

We can decompose the fields into a z component, and in-plane components (which are in the xy plane, denoted by the subscript $\parallel$). Now,

$$\vec{E}_{\parallel} = \vec{E} - \hat{e}_z\, E_z = (\hat{e}_z \times \vec{E}) \times \hat{e}_z = -\hat{e}_z \times (\hat{e}_z \times \vec{E}_{\parallel}) \,, \tag{4.14a}$$

$$\vec{B}_{\parallel} = \vec{B} - \hat{e}_z\, B_z = (\hat{e}_z \times B) \times \hat{e}_z = -\hat{e}_z \times (\hat{e}_z \times B_{\parallel}) \,. \tag{4.14b}$$

The $\hat{e}_z \times (\hat{e}_z \times \ldots)$-operation thus amounts to a simple minus sign for transverse vectors. We can thus decompose Faraday's law into parallel and transverse components. We start from the relation

$$\vec{\nabla} \times \vec{E} = \left[\hat{e}_z \nabla_z + \vec{\nabla}_\parallel\right] \times \left[\hat{e}_z\, E_z + \vec{E}_\parallel\right] = \mathrm{i}\,\omega\, \vec{B}(\vec{r})\,. \tag{4.15}$$

Then, in view of $\hat{e}_z \times \hat{e}_z = \vec{0}$,

$$\vec{\nabla} \times \vec{E} = \underbrace{\hat{e}_z \nabla_z \times \vec{E}_\parallel - \hat{e}_z \times \vec{\nabla}_\parallel E_z}_{\text{in-plane}} + \underbrace{\vec{\nabla}_\parallel \times \vec{E}_\parallel}_{z\text{-oriented}} = \underbrace{\mathrm{i}\omega\, \vec{B}_\parallel}_{\text{in-plane}} + \underbrace{\mathrm{i}\omega\, \hat{e}_z\, B_z}_{z\text{-oriented}}\,, \tag{4.16}$$

$$\vec{\nabla} \times \vec{B} = \underbrace{\hat{e}_z\, \nabla_z \times \vec{B}_\parallel - \hat{e}_z \times \vec{\nabla}_\parallel B_z}_{\text{in-plane}} + \underbrace{\vec{\nabla}_\parallel \times \vec{B}_\parallel}_{z\text{-oriented}} = \underbrace{-\,\mathrm{i}\,\frac{\omega}{c^2}\,\epsilon_r\,\mu_r\,\vec{E}_\parallel}_{\text{in-plane}} \underbrace{-\,\mathrm{i}\,\frac{\omega}{c^2}\,\epsilon_r\,\mu_r\,\hat{e}_z\,E_z}_{z\text{-oriented}}\,. \tag{4.17}$$

Thus, separating this expression into z-oriented and in-plane components, we can immediately write down a number of relations. The in-plane component of Eq. (4.16) is

$$\hat{e}_z \times \nabla_z\, \vec{E}_\parallel - \hat{e}_z \times \vec{\nabla}_\parallel E_z = \mathrm{i}\,\omega\,\vec{B}_\parallel\,. \tag{4.18}$$

We now apply the $(\hat{e}_z \times \ldots)$-operation to both sides and in view of Eq. (4.14), we obtain

$$\nabla_z \vec{E}_\parallel + \mathrm{i}\omega \left(\hat{e}_z \times \vec{B}_\parallel\right) = \vec{\nabla}_\parallel E_z\,, \tag{4.19}$$

We can rewrite Eq. (4.19), replacing $\nabla_z \to \mathrm{i}\,k$,

$$\boxed{\textbf{Input Equation 1:}} \qquad \mathrm{i}\,k\,\vec{E}_\parallel = \vec{\nabla}_\parallel\, E_z - \mathrm{i}\omega(\hat{e}_z \times \vec{B}_\parallel)\,. \tag{4.20}$$

For reference, the z-oriented component of Eq. (4.16) is

$$\vec{\nabla}_\parallel \times \vec{E}_\parallel = \mathrm{i}\omega\,\hat{e}_z\,B_z\,, \qquad \hat{e}_z \cdot \left(\vec{\nabla}_\parallel \times \vec{E}_\parallel\right) = \mathrm{i}\omega\,B_z\,. \tag{4.21}$$

Analogously, the in-plane component of Eq. (4.17) reads

$$\hat{e}_z \times \nabla_z \vec{B}_\parallel - \hat{e}_z \times \vec{\nabla}_\parallel B_z = -\mathrm{i}\,\frac{\omega}{c^2}\,\epsilon_r\,\mu_r\,\vec{E}_\parallel\,. \tag{4.22}$$

If we again apply the $(\hat{e}_z \times \ldots)$-operation to both sides and use Eq. (4.14), we obtain

$$\nabla_z \vec{B}_\parallel - \mathrm{i}\frac{\omega}{c^2}\,\epsilon_r\,\mu_r\,\left(\hat{e}_z \times \vec{E}_\parallel\right) = \vec{\nabla}_\parallel B_z\,, \tag{4.23}$$

so that, in particular, replacing $\nabla_z \to \mathrm{i}\,k$,

$$\boxed{\textbf{Input Equation 2:}} \qquad \mathrm{i}k\,\vec{B}_\parallel - \mathrm{i}\frac{\omega}{c^2}\,\epsilon_r\,\mu_r\,\left(\hat{e}_z \times \vec{E}_\parallel\right) = \vec{\nabla}_\parallel B_z\,. \tag{4.24}$$

If we use the in-plane component of Eq. (4.17) directly, replacing ∇_z by $\mathrm{i}\,k$, then we can write

$$\boxed{\textbf{Input Equation 3:}} \qquad \hat{e}_z \times \vec{B}_\parallel = \frac{1}{\mathrm{i}k}(\hat{e}_z \times \vec{\nabla}_\parallel B_z) - \frac{\omega\,\epsilon_r\,\mu_r}{c^2\,k}\vec{E}_\parallel\,. \tag{4.25}$$

For reference, the in-plane component of Eq. (4.17) says that

$$\hat{e}_z \cdot \left(\vec{\nabla}_\parallel \times \vec{B}_\parallel\right) = -\mathrm{i}\,\frac{\omega}{c^2}\,\epsilon_r\,\mu_r\,E_z\,. \tag{4.26}$$

Also, since the divergence of both electric and magnetic fields vanishes, one has, trivially,

$$\vec{\nabla}_\parallel \cdot \vec{E}_\parallel + \nabla_z\,E_z = 0\,, \qquad \vec{\nabla}_\parallel \cdot \vec{B}_\parallel + \nabla_z\,B_z = 0\,. \tag{4.27}$$

Finally, we solve for $\vec{E}_{\|}$ and $\vec{B}_{\|}$ if E_z and B_z are known (and not both are zero). Using the result from Eq. (4.25) (input equation 3) in Eq. (4.20) (input equation 1), we have

$$\mathrm{i}k\vec{E}_{\|} = \vec{\nabla}_{\|} E_z - \frac{\omega}{k}\left(\hat{\mathrm{e}}_z \times \vec{\nabla}_{\|} B_z\right) + \mathrm{i}\,\frac{\omega^2\,\epsilon_r\,\mu_r}{c^2 k}\,\vec{E}_{\|}\,. \tag{4.28}$$

From the last equation, multiplying by $\mathrm{i}\,k$, we find that

$$\left(\frac{\omega^2}{c^2}\epsilon_r\mu_r - k^2\right)\,\vec{E}_{\|} = \mathrm{i}\,k\,\vec{\nabla}_{\|} E_z - \mathrm{i}\,\omega\,(\hat{\mathrm{e}}_z \times \vec{\nabla}_{\|} B_z)\,. \tag{4.29}$$

We can finally solve for $\vec{E}_{\|}$,

$$\boxed{\textbf{\textit{E}-Field from \textit{z} Components:}}\quad \vec{E}_{\|} = \mathrm{i}\,\frac{1}{\omega^2\,\epsilon_r\,\mu_r/c^2 - k^2}\,\left(k\,\vec{\nabla}_{\|} E_z - \omega\,(\hat{\mathrm{e}}_z \times \vec{\nabla}_{\|})B_z\right)\,. \tag{4.30}$$

In Eq. (4.24) (input equation 2), we can replace $\vec{E}_{\|}$ with the result just obtained and obtain

$$\boxed{\textbf{\textit{B}-Field from \textit{z} Components:}}\quad \vec{B}_{\|} = \mathrm{i}\,\frac{1}{\omega^2\,\epsilon_r\,\mu_r/c^2 - k^2}\,\left(k\,\vec{\nabla}_{\|} B_z + \frac{\omega\,\epsilon_r\,\mu_r}{c^2}\,\left(\hat{\mathrm{e}}_z \times \vec{\nabla}_{\|}\right)E_z\right)\,. \tag{4.31}$$

We have discussed waves which manifestly travel in the positive z direction. For waves traveling in the opposite direction, one just changes k to $-k$. We should also point out that in vacuum, we have $\epsilon_r \to 1$ and $\mu_r \to 1$, so that the denominator $\omega^2\,\epsilon_r\,\mu_r/c^2 - k^2$ vanishes. However, in this case, the z components of the electric and magnetic fields also vanish; we have assumed that the wave propagates in the z direction, and the magnetic and electric fields are strictly transverse in vacuum.

Equations (4.30) and (4.31) are valid for the general propagation of waves in a medium with permittivity ϵ_r and permeability μ_r, not necessarily for a waveguide. The waveguide aspect comes in through the boundary conditions, which will be discussed next.

4.2.2 Boundary Conditions at the Surface

General Considerations. We first review some of the general considerations for boundary conditions at the boundaries of two materials, before specializing them for the cases of perfect conductors, wave guides and perfectly conducting cavities. For the magnetic field, in view of $\vec{\nabla}\cdot\vec{B} = 0$, one has with the perpendicular component $B_\perp$ sticking out of the plane,

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow [(B_\perp)_1 - (B_\perp)_2]\,\Delta s\,\Delta w = 0 \tag{4.32}$$

and so

$$(B_\perp)_1 = (B_\perp)_2\,. \tag{4.33}$$

For the dielectric displacement, one has

$$\vec{\nabla}\cdot\vec{D} = \rho \Rightarrow [(D_\perp)_1 - (D_\perp)_2]\,\Delta s\,\Delta w = \rho\,\Delta s\,\Delta w\,\Delta z = \Delta q\,, \tag{4.34}$$

and so

$$(D_\perp)_1 - (D_\perp)_2 = \frac{\Delta q}{\Delta s\,\Delta w} = \sigma\,, \tag{4.35}$$

where σ is the surface charge density (charge per unit area). For the electric field, one has

$$\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t} \Rightarrow \left[(E_{\|})_1 - (E_{\|})_2\right]\,\Delta\ell = -\frac{\partial}{\partial t}\,B\,\Delta\ell\,\Delta z\,, \tag{4.36}$$

and so, since $\partial B/\partial t$ cannot become singular,

$$(E_\parallel)_1 = (E_\parallel)_2\,. \tag{4.37}$$

Let us lay our coordinate system so that the xy plane is the separation plane of the two components. If the parallel component vanishes, then it means that both the x as well as the y components must vanish. This can be formulated as a vector equation, which for arbitrary orientation of the surface normal $\hat{n}$ reads

$$\hat{n} \times \left(\vec{E}_1 - \vec{E}_2\right) = 0\,. \tag{4.38}$$

For the magnetic field $\vec{H}$, one has

$$\vec{\nabla} \times \vec{H} = \frac{\partial}{\partial t}\,\vec{D} + \vec{J} \Rightarrow \left[(H_\parallel)_1 - (H_\parallel)_2\right]\,\Delta\ell = \frac{\partial D}{\partial t}\,\Delta\ell\,\Delta z + J\,\Delta\ell\,\Delta z\,. \tag{4.39}$$

Now, since $\partial D/\partial t$ cannot become singular,

$$\left[(H_\parallel)_1 - (H_\parallel)_2\right] = J\,\Delta z \equiv K\,. \tag{4.40}$$

The question then is whether $J\Delta z$ can attain a finite value as one crosses the boundary layer. The answer is yes. Consider a perfect conductor carrying a current I transported along the outer layer of the conductor. Since the current inside the perfect conductor vanishes, and assuming that the current is distributed uniformly along the outer layer of a circular conductor of radius R, we have

$$J\Delta z = \frac{\Delta i}{\Delta\ell\Delta z}\,\Delta z = \frac{\Delta i}{\Delta\ell} = \frac{I}{2\pi R} = \text{finite}\,. \tag{4.41}$$

Here, Δi is an infinitesimal current element. In general, we have

$$\hat{n} \times \left[\vec{H}_1 - \vec{H}_2\right] = \vec{K}\,. \tag{4.42}$$

In summary, the existence of surface charge densities, σ, and surface current densities, $\vec{K}$, at the interface provide the following boundary conditions:

$$\hat{n} \cdot (\vec{D}_1 - \vec{D}_2) = \sigma\,, \tag{4.43a}$$

$$\hat{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0\,, \tag{4.43b}$$

$$\hat{n} \times (\vec{E}_1 - \vec{E}_2) = \vec{0}\,, \tag{4.43c}$$

$$\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{K}\,. \tag{4.43d}$$

Application to the Wave Guide/Perfect Conductor. We consider the specialization of the general boundary conditions onto the case of a perfect electric conductor. The walls of a wave guide are typically idealized as such perfectly conducting walls. Inside the conductor, the electric field is zero, because any conceivably entering field is immediately compensated by a rearrangement of conducting electrons. Hence,

$$\boxed{\textbf{Perfect Conductor I:}} \qquad \hat{n} \times \vec{E}\big|_S = 0\,, \qquad \text{or} \qquad E_{\parallel,S} = 0\,, \tag{4.44}$$

where $\hat{n}$ is the surface normal and the subscript S denotes the evaluation on a point (x, y, z) that satisfies the equation $F(x, y, z) = 0$ defining the surface of the conductor. Equation (4.44) is equivalent to (4.43c) in the limit $\vec{E}_2 \to \vec{0}$. We use the notation $E_{\parallel,S}$ in order to denote the component of $\vec{E}$ parallel and inside the surface.

There is a further boundary condition, which concerns the magnetic field. First, we observe that in principle, there is a no obstacle against a static magnetic field entering a perfect conductor. However, let us suppose that a time varying, oscillating field enters the conductor. Then, in view of the Faraday law,

$$\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t} \vec{B} = 0 \,, \tag{4.45}$$

one would generate a time-varying electric field in the conductor, which in turn cannot exist. One might also invoke the Ampere–Maxwell law for an alternative derivation. Specifically, let us suppose that $\hat{n}$ is the surface normal of the conductor. The magnetic field is an oscillating field by assumption. If the projection of the magnetic field on the surface normal $\hat{n} \cdot \vec{B}\big|_S$ were nonzero, we would generate a time-varying electric field inside the plane defining the surface. This would be contradictory. Hence,

$$\boxed{\textbf{Perfect Conductor II:}} \qquad \hat{n} \cdot \vec{B}\big|_S = 0 \,, \qquad \text{or} \qquad B_{\perp,S} = 0 \,. \tag{4.46}$$

However, a nonvanishing and oscillating parallel component of the magnetic field may exist in the vicinity of the conductor. By the Faraday law, this oscillating magnetic field gives rise to an oscillating electric field, perpendicular to the surface, and this is not in contradiction to the boundary condition for the electric field.

If we assume the wave guide to be aligned along the z axis, then for a rectangular wave guide the boundaries are the xy or the yz planes. There are two kinds of propagating electromagnetic modes for which the boundary conditions (4.44) and (4.46) are fulfilled. These are called the transverse magnetic (TM) and transverse electric (TE) modes, and we shall discuss them in detail. The fact that the TM and TE modes actually fulfill Eqs. (4.44) and (4.46) is tied to the way in which the perpendicular/transverse ($\|$) components of the fields are calculated from the z components, see Eqs. (4.30) and (4.31).

From Eqs. (4.30) and (4.31), it is clear that any fields inside the waveguide can be decomposed into those generated for a vanishing field $E_z = 0$ (in which case $\vec{E} = \vec{E}_\|$, and the electric field is transverse), and those generated for a vanishing field $B_z = 0$ (in which case $\vec{B} = \vec{B}_\|$, and the magnetic field is transverse). These are naturally referred to as the transverse electric (TE) and transverse magnetic (TM) modes, respectively.

We shall discuss boundary conditions for both TE and TM modes in the following. The first and most straightforward case concerns the TM mode,

$$\boxed{\textbf{TM defining property:}} \qquad B_z = 0 \qquad \text{everywhere} \,, \qquad \vec{B} = \vec{B}_\| \,. \tag{4.47}$$

For TM modes, we also have to require that according to Eq. (4.44), the component of the electric field tangent to the interface (E_z) must vanish at the surface, i.e., $E_z\big|_S = 0$. For TM modes, we have according to Eq. (4.31),

$$\boxed{\textbf{TM additional property:}} \qquad \vec{B}_\| \propto \hat{\text{e}}_z \times \vec{\nabla}_\| E_z \,, \tag{4.48}$$

At the boundary, since $E_z\big|_S = 0$ by assumption, the vector $\vec{\nabla}_\| E_z$ (and also $\vec{\nabla} E_z$) is perpendicular to the surface. Forming the vector product of $\vec{\nabla}_\| E_z$ with $\hat{\text{e}}_z$ at the surface, we thus get a vector lying inside the surface, i.e., we have shown that $\vec{B}_\|$ must necessarily be parallel to the surface in its immediate vicinity. We have thus shown once more that it fulfills the boundary condition (4.46). We summarize

$$\boxed{\textbf{TM properties:}} \qquad B_z = 0 \qquad \text{everywhere} \,, \qquad E_z\big|_S = 0 \,. \tag{4.49}$$

The second possibility is given by the TE mode. In that case,

$$\boxed{\textbf{TE defining property:}} \qquad E_z = 0 \qquad \text{everywhere} \,, \qquad \vec{E} = \vec{E}_\| \,, \tag{4.50}$$

and the boundary condition (4.44) for the electric field is automatically fulfilled. We now take advantage of Eq. (4.31) and establish that $\vec{B}_{\parallel} \propto \vec{\nabla}_{\parallel} B_z$. Thus, the normal component $\hat{n} \cdot \vec{B}_{\parallel} = \hat{n} \cdot \vec{B}$ can alternatively be obtained by projecting $\vec{\nabla}_{\parallel} B_z$ onto the surface normal. The corresponding condition on B_z is

$$\hat{n} \cdot \vec{B} = \hat{n} \cdot \vec{B}_{\parallel} \propto \left(\hat{n} \cdot \vec{\nabla}_{\parallel}\right) B_z = \left(\hat{n} \cdot \vec{\nabla}\right) B_z \equiv \frac{\partial B_z}{\partial n} \, . \tag{4.51}$$

At the surface, however, the projection $\hat{n} \cdot \vec{B}$ needs to vanish, and so

$$0 = \hat{n} \cdot \vec{B}\big|_S \propto \hat{n} \cdot \vec{B}_{\parallel}\big|_S \propto \left(\hat{n} \cdot \vec{\nabla}_{\parallel}\right) B_z\Big|_S = \left(\hat{n} \cdot \vec{\nabla}\right) B_z\Big|_S \equiv \left.\frac{\partial B_z}{\partial n}\right|_S \, . \tag{4.52}$$

The latter expression merely is a definition motivated by the relation

$$\frac{\partial B_z(\vec{r} + \xi\,\hat{n})}{\partial \xi} = \hat{n} \cdot \vec{\nabla} B_z = \hat{n} \cdot \vec{\nabla}_{\parallel} B_z \, . \tag{4.53}$$

We summarize

$$\boxed{\textbf{TE properties:}} \qquad E_z = 0 \qquad \text{everywhere}\,, \qquad \left.\frac{\partial B_z}{\partial n}\right|_S = 0 \, . \tag{4.54}$$

Finally, in view of Eqs. (4.30), (4.31) and (4.49), we find for the transverse magnetic (TM) modes,

$$\boxed{\textbf{TM wave:}} \qquad B_z = 0 \quad \text{everywhere and} \quad E_z\big|_S = 0\,, \tag{4.55a}$$

$$\vec{E}_{\parallel} = \mathrm{i}\, \frac{1}{\epsilon_r\, \mu_r\, \omega^2/c^2 - k^2}\, k\, \vec{\nabla}_{\parallel} E_z\,, \tag{4.55b}$$

$$\vec{B}_{\parallel} = \mathrm{i}\, \frac{1}{\epsilon_r\, \mu_r\, \omega^2/c^2 - k^2}\, \omega\, \epsilon_r\, \mu_r\, \left(\hat{e}_z \times \vec{\nabla}_{\parallel} E_z\right)\,, \tag{4.55c}$$

$$0 = \left(\vec{\nabla}_{\parallel}^2 + \epsilon_r\, \mu_r\, \frac{\omega^2}{c^2} - k^2\right)\, E_z(x,y)\,. \tag{4.55d}$$

In view of Eqs. (4.30), (4.31) and (4.54), we find for the transverse electric (TE) modes,

$$\boxed{\textbf{TE wave:}} \qquad E_z = 0 \quad \text{everywhere and} \quad \left.\frac{\partial B_z}{\partial n}\right|_S = 0\,, \tag{4.56a}$$

$$\vec{B}_{\parallel} = \mathrm{i} \frac{1}{\epsilon_r\, \mu_r\, \omega^2/c^2 - k^2}\, k\, \vec{\nabla}_{\parallel} B_z\,, \tag{4.56b}$$

$$\vec{E}_{\parallel} = \mathrm{i} \frac{1}{\epsilon_r\, \mu_r\, \omega^2/c^2 - k^2}\, \left[-\omega \left(\hat{e}_z \times \vec{\nabla}_{\parallel} B_z\right)\right]\,, \tag{4.56c}$$

$$0 = \left(\vec{\nabla}_{\parallel}^2 + \epsilon_r \mu_r\, \frac{\omega^2}{c^2} - k^2\right)\, B_z(x,y)\,. \tag{4.56d}$$

The boundary conditions give rise to eigenvalues of k (dependent on ω) for which the propagation is allowed. These eigenvalues may be continuous. Since the boundary conditions for E_z and B_z are different, the eigenvalues are also different for TE as opposed to TM modes. The allowed TE and TM waves (and the TEM wave, if it exists) provide a complete set of waves from which one can construct an arbitrary electromagnetic disturbance in the waveguide or cavity.

Subtle Point. Up to now, we have denoted the perpendicular component(s) of the electric field as $\vec{E}_{\parallel}$. This component is perpendicular to the propagation direction (z axis) and provides for the components perpendicular to the boundary of the wave guide along the walls which are parallel to the z axis. For a cavity,

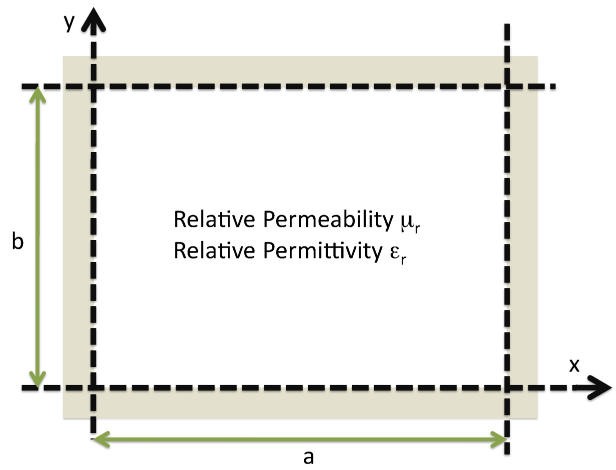


Figure 4.1: Picture of the rectangular waveguide.

however, $\vec{E}_\parallel$ is (by convention usually identified as the) *parallel* to the endcap surfaces at $z = 0$ and $z = d$ (if the z dimension of the cavity is d). Similarly, $\vec{B}_\parallel$ also is *parallel* to the endcap surfaces at $z = 0$ and $z = d$. From this consideration, one may derive further conditions on the endcap surface, e.g., in view of Eq. (4.56c),

$$\vec{E}_\parallel \propto \hat{e}_z \times \vec{\nabla}_\parallel B_z = \hat{e}_z \times \left(\hat{e}_x \frac{\partial}{\partial x} B_z + \hat{e}_y \frac{\partial}{\partial y} B_z \right) = \hat{e}_y \frac{\partial}{\partial x} B_z - \hat{e}_x \frac{\partial}{\partial y} B_z \,. \tag{4.57}$$

Along the endcap surfaces, both partial derivatives thus have to vanish.

4.2.3 Modes in a Rectangular Waveguide

We determine the TE modes in a rectangular waveguide with dimensions a in the x direction and b in the y direction (with $a > b$), and $0 < x < a$, and $0 < y < b$. Note that this means that $E_z = 0$ everywhere. $\vec{E}$ is thus transverse, and we talk about the transverse electric (TE) modes. So, $\vec{E}_\parallel$ will be found from B_z alone, according to the relation

$$\begin{aligned} \vec{E}_\parallel &= \mathrm{i} \, \frac{1}{\epsilon_r \, \mu_r \, \omega^2/c^2 - k^2} \, \left[-\omega (\hat{e}_z \times \vec{\nabla}_\parallel) B_z \right] \\ &= -\frac{\omega}{k} \, \hat{e}_z \times \left(\mathrm{i} \, \frac{k}{\epsilon_r \, \mu_r \, \omega^2/c^2 - k^2} \, \vec{\nabla}_\parallel B_z \right) = -\frac{\omega}{k} \, \hat{e}_z \times \vec{B}_\parallel \,. \end{aligned} \tag{4.58}$$

This, in particular, means that $\vec{E}_\parallel$ and $\vec{B}_\parallel$ are still perpendicular to each other. In order to solve the wave equation for B_z,

$$\left(\vec{\nabla}_\parallel^2 + \epsilon_r \, \mu_r \, \frac{\omega^2}{c^2} - k^2 \right) \, B_z(x, y) = 0 \,. \tag{4.59}$$

The general solution is:

$$B_z(x, y) = C_1 \, \mathrm{e}^{+\mathrm{i}\vec{k}\cdot\vec{r}_\parallel} + C_2 \, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{r}_\parallel} \,, \qquad \vec{r}_\parallel = \hat{e}_x \, x + \hat{e}_y \, y \,. \tag{4.60}$$

The form for $B_z(x, y)$ which is non-zero when $x = y = 0$ is

$$B_z(x, y) = B_0 \, \cos(k_x \, x) \, \cos(k_y \, y) \,. \tag{4.61}$$

Transverse Electric Mode $\mathrm{TE}_{1,0}$

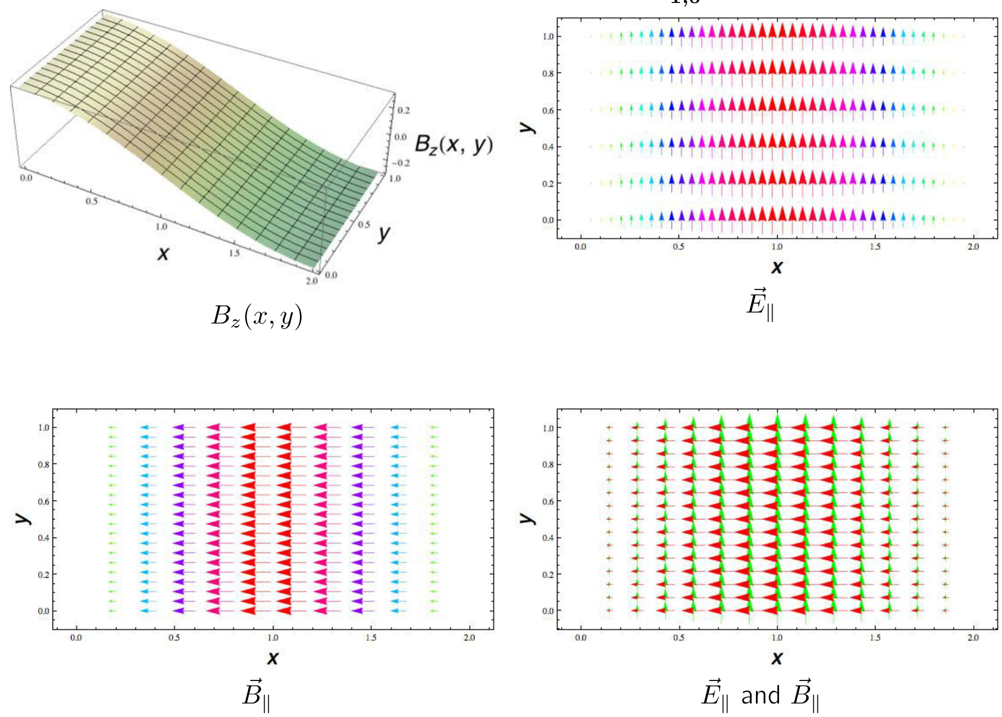


Figure 4.2: Four plots illustrating the transverse electric mode $\mathrm{TE}_{1,0}$. The leftmost upper plot shows $B_z(x,y)$ as a function of x and y. As the mode is transverse, $E_z(x,y)$ vanishes. The rightmost upper plot shows, as a vector plot, $\vec{E}_{\parallel}(x,y)$ in a plane of constant z, at a particular time of the sinusoidal oscillation. All vector arrows oscillate sinusoidally. The left lower plot shows $\vec{B}_{\parallel}(x,y)$ in a plane of constant z, at a particular time of the sinusoidal oscillation. It is clearly visible that $(\hat{n} \cdot \vec{\nabla}_{\parallel}) B_z$ vanishes at the surface, as for all TE modes. Finally, in the right lower plot, the green arrows indicate the electric field, whereas the red ones indicate the magnetic field, illustrating the perpendicular character of the two fields.

We recall the boundary conditions for transverse electric (TE) modes in the form [see also Eq. (4.52)],

$$\left(\hat{n} \cdot \vec{\nabla}_{\parallel}\right) B_z \bigg|_S = 0 \,, \tag{4.62a}$$

$$\frac{\partial}{\partial x} B_0 \, \cos(k_x \, x) \, \cos(k_y \, y) \bigg|_{x=0,a} = 0 \,, \tag{4.62b}$$

$$\frac{\partial}{\partial y} B_0 \, \cos(k_x \, x) \, \cos(k_y \, y) \bigg|_{y=0,b} = 0 \,. \tag{4.62c}$$

Any presence of a sine function in these two equations would lead to a nonvanishing derivative unless B_0 vanishes and thus the whole term. This justifies, a posteriori, our an ansatz with a cosine. Then,

$$\sin(k_x \, a) = \sin(k_x \, 0) = 0 \,, \qquad k_x = \frac{m\pi}{a} \,, \qquad m = 0, 1, 2, \dots \,, \tag{4.63a}$$

$$\sin(k_y \, b) = \sin(k_y \, 0) = 0 \,, \qquad k_y = \frac{n\pi}{b} \,, \qquad n = 0, 1, 2, \dots \,. \tag{4.63b}$$

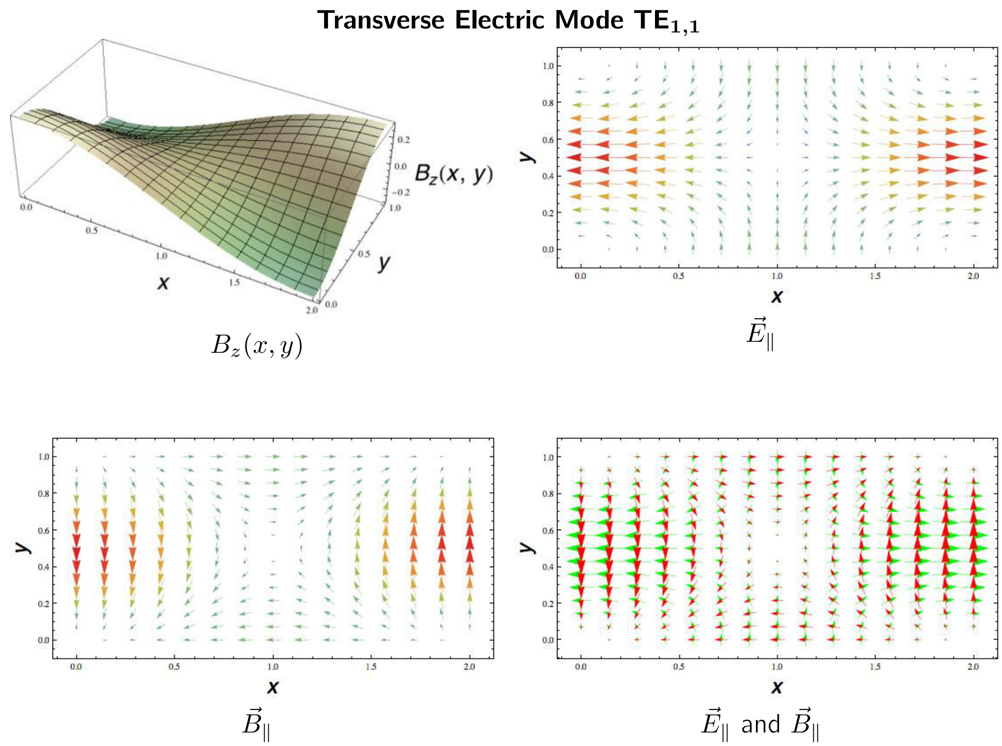


Figure 4.3: Same as Fig. 4.2, for the transverse electric mode $\mathrm{TE}_{1,1}$.

This means that for given m and n, a dispersion relation can be established between ω and k,

$$B_z(x,y) = B_{0,mn}\ \cos\left(\frac{m\pi x}{a}\right)\ \cos\left(\frac{n\pi y}{b}\right)\,, \tag{4.64}$$

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \epsilon_r\,\mu_r\,\frac{\omega^2}{c^2} - k^2\,, \qquad m,n = 0,1,2,3,\dots\,, \tag{4.65}$$

$$k^2 = \epsilon_r\,\mu_r\,\frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 > 0\,, \qquad k = \sqrt{\epsilon_r\,\mu_r\,\frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}\,. \tag{4.66}$$

Propagation in the (m,n) mode cannot proceed unless the solution for k is real,

$$\omega > \omega_{mn} \equiv \frac{c}{\sqrt{\epsilon_r\mu_r}}\ \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2} = \frac{\pi c}{\sqrt{\epsilon_r\mu_r}}\ \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2}\,. \tag{4.67}$$

for $\omega < \omega_{\min}$, the wave number k becomes purely imaginary, $k = \mathrm{i}\,|k|$, and the wave, propagating in the z direction, is proportional to $\exp(-|k|\,z)$, becomes evanescent. Moreover, ω_{mn} is the m and n-dependent waveguide angular frequency. The minimum frequency is obtained for

$$\omega_{\min} = \frac{\pi}{\sqrt{\epsilon_r\mu_r}}\,\frac{c}{a}\,, \qquad m = 1, \quad n = 0, \qquad a > b\,. \tag{4.68}$$

In view of the occurrence of the factor c/a, the minimum frequency for propagation in the waveguide is commensurate with the inverse time it takes light to travel the spatial dimension of the waveguide. For a

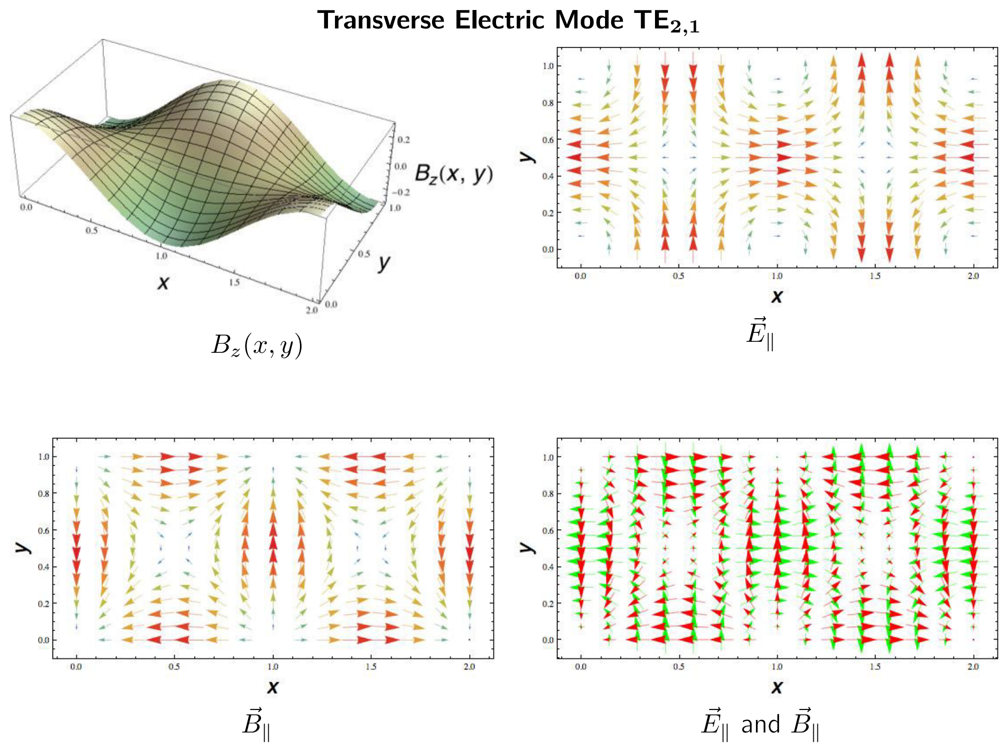


Figure 4.4: Same as Figs. 4.2 and 4.3, for the transverse electric mode $\mathrm{TE}_{2,1}$.

non-trivial solution, m and n cannot both be zero. When the condition (4.67) is fulfilled, real rather than imaginary solutions can be found for the wave vector. The full solution for each TE_{mn} mode is:

$$\boxed{\textbf{TE:}} \quad B_z(x,y) = B_{z,mn}(x,y) = B_{0,mn}\, \cos\left(\frac{m\pi x}{a}\right)\, \cos\left(\frac{n\pi y}{b}\right), \qquad E_z(x,y) = 0\,, \tag{4.69a}$$

$$\vec{B}_\parallel(m,n) = \frac{\mathrm{i}\,k}{\omega^2\,\epsilon_r\,\mu_r/c^2 - k^2}\,\vec{\nabla}_\parallel B_{z,mn}\,\mathrm{e}^{\mathrm{i}\,(k\,z-\omega\,t)} = -\mathrm{i}\,k\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{-1} B_{0,mn} \tag{4.69b}$$
$$\times\left[\hat{\mathrm{e}}_x \frac{m\pi}{a}\,\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) + \hat{\mathrm{e}}_y\,\frac{n\pi}{b}\,\cos\left(\frac{m\pi x}{a}\right)\,\sin\left(\frac{n\pi y}{b}\right)\right]\,\mathrm{e}^{\mathrm{i}(k\,z-\omega\,t)}\,,$$

$$\vec{E}_\parallel(m,n) = -\,\frac{\omega}{k}\,\left[\hat{\mathrm{e}}_z\times\vec{B}_\parallel(m,n)\right] = -\mathrm{i}\,\omega\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{-1} B_{0,mn} \tag{4.69c}$$
$$\times\left[\hat{\mathrm{e}}_x \frac{n\pi}{b}\,\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) - \hat{\mathrm{e}}_y\,\frac{m\pi}{a}\,\sin\left(\frac{m\pi x}{a}\right)\,\cos\left(\frac{n\pi y}{b}\right)\right]\,\mathrm{e}^{\mathrm{i}(k\,z-\omega\,t)}\,,$$

$$k^2 = \epsilon_r\,\mu_r\,\frac{\omega^2}{c^2} - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]. \tag{4.69d}$$

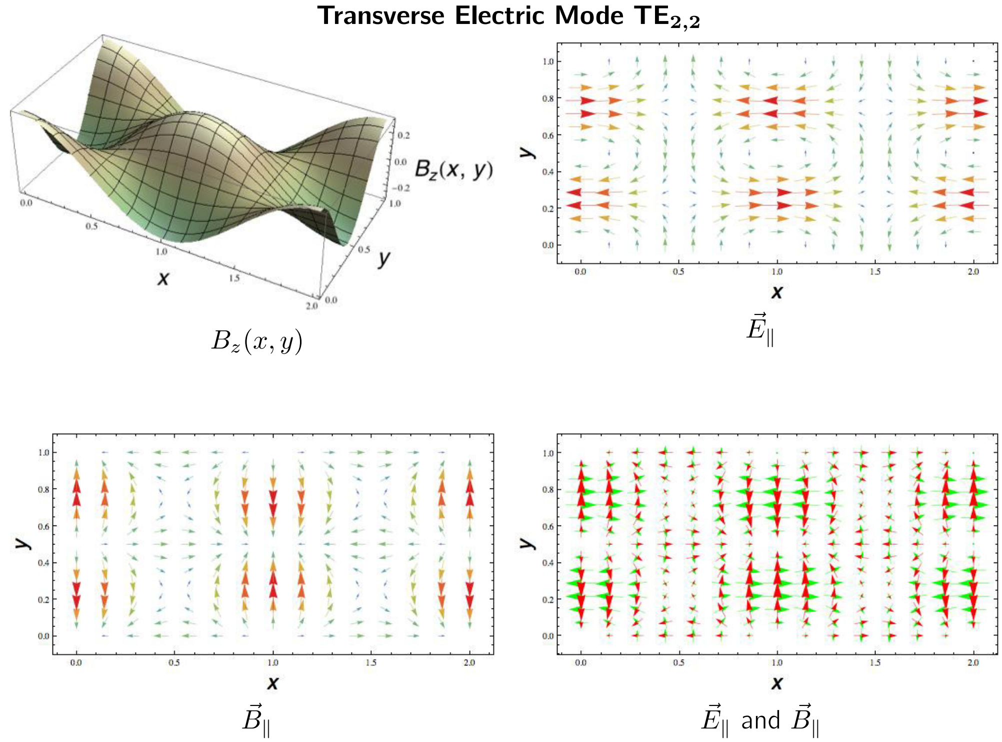


Figure 4.5: Same as Figs. 4.2, 4.4, and 4.5, but for the transverse electric mode $TE_{2,2}$.

Restoring the time dependence, the solution for the $TE_{1,0}$ mode with $m = 1$ and $n = 0$ is found to be

$$B_z(x,y) = B_0 \, \cos\left(\frac{\pi x}{a}\right) \, \mathrm{e}^{\mathrm{i}\,(k\,z-\omega\,t)}\,, \tag{4.70a}$$

$$\vec{B}_\parallel(m=1,n=0) = -\,\mathrm{i}\,\frac{ka}{\pi}\,B_0\,\hat{\mathrm{e}}_x\,\sin\left(\frac{\pi x}{a}\right)\,\mathrm{e}^{\mathrm{i}\,(k\,z-\omega\,t)}\,, \tag{4.70b}$$

$$\vec{E}_\parallel(m=1,n=0) = \mathrm{i}\,\frac{\omega a}{\pi}\,B_0\,\hat{\mathrm{e}}_y\,\sin\left(\frac{\pi x}{a}\right)\,\mathrm{e}^{\mathrm{i}\,(k\,z-\omega\,t)}\,, \tag{4.70c}$$

$$k = k_{10} = \sqrt{\epsilon_r\mu_r\,\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2} = \sqrt{\epsilon_r\mu_r}\,\frac{\omega}{c}\sqrt{1-\frac{1}{\epsilon_r\mu_r}\,\left(\frac{\pi c}{\omega a}\right)^2}\,. \tag{4.70d}$$

We recall that propagation happens only for

$$\omega^2 > \omega_{\min} = \frac{\pi}{\sqrt{\epsilon_r\mu_r}}\,\frac{c}{a}\,. \tag{4.71}$$

Note the 90° phase difference between B_x and B_z arising from the $-\mathrm{i} = e^{-\mathrm{i}\pi/2}$ factor. The perpendicular components $\vec{B}_\parallel$ and $\vec{E}_\parallel$ are 180° out of phase. Graphical representations can be found in Figs. 4.2, 4.3, 4.4 and 4.5. In many cases, one designs the wave guide so that the $m = 1$, $n = 0$ mode is the dominant TE

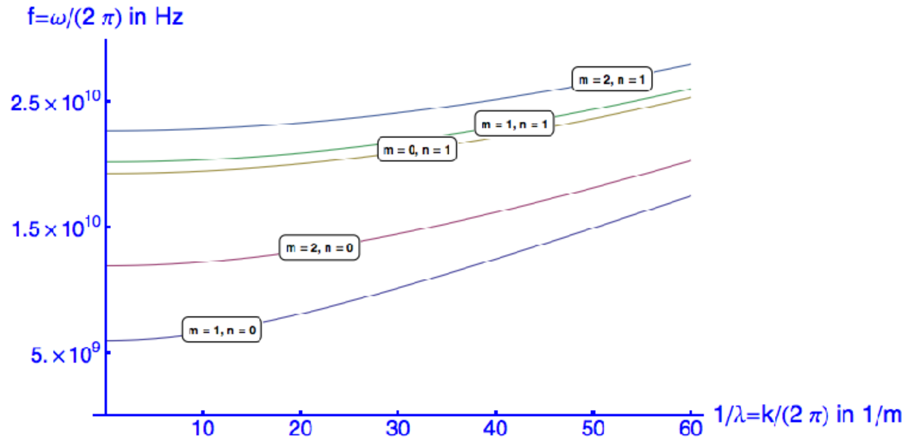


Figure 4.6: Dispersion relations for the first few modes of a rectangular waveguide, for all modes with $m \leq 2$ and $n \leq 2$. The dimensions of the waveguide are $a = 2.3\,\mathrm{cm}$ and $b = 0.71\,\mathrm{cm}$. The relative permittivity of the waveguide medium is $\epsilon_r = 1.2$, and the permeability is $\mu_r = 1.0$, independent of the angular frequency ω. The abscissa gives the inverse wavelength $1/\lambda$ in units of inverse meter (m), where $1/\lambda = k/(2\pi)$, and the ordinate axis gives the light frequency $f = \omega/(2\pi)$ in units of cycles per second, which is equal to Hertz (Hz). We note that the SI mksA unit of the angular frequency ω is radians per second. A full oscillation period per second corresponds to one cycle per second, or 2π radians per second $(\mathrm{rad/s})$.

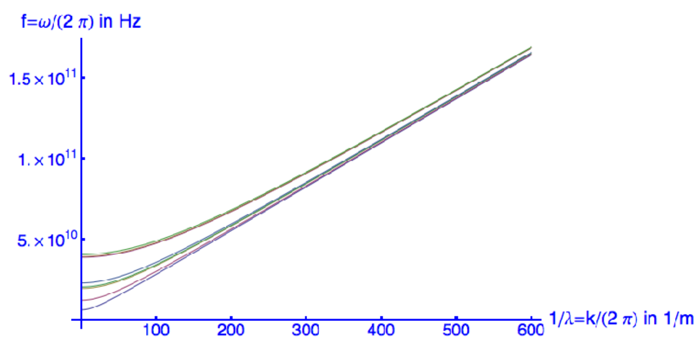


Figure 4.7: Same sa Fig. 4.6, but for a region of medium wave number, where the modes of the wave guide approach the free dispersion relation $\omega = ck/\sqrt{\epsilon_r\,\mu_r}$.

mode. The dispersion relation can also be written as

$$k = \frac{\sqrt{\epsilon_r \mu_r}}{c}\;\sqrt{\omega^2 - \omega_{mn}^2}\,, \tag{4.72}$$

$$\omega_{mn} = \frac{\pi c}{\sqrt{\epsilon_r \mu_r}}\;\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2}\,. \tag{4.73}$$

For each mode, the k_{mn} varies with frequency $\omega > \omega_{mn}$. The ω_{mn} is the cutoff frequency for the mode. The dispersion relations in Fig. 4.6, 4.7 and 4.8 represent the functional relationships $\omega = \omega(k)$ for the first few modes of a typical waveguide.

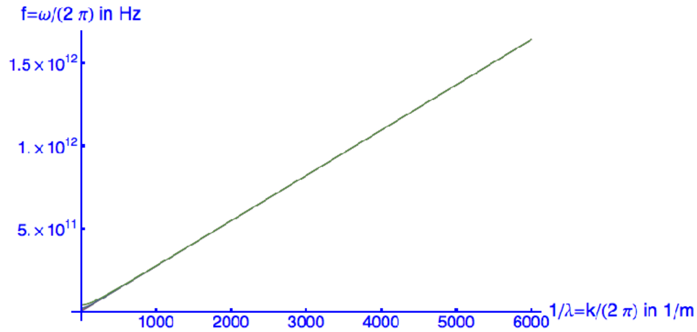


Figure 4.8: Same as Figs. 4.6 and 4.7, but for even larger wave numbers. The deviation from the dispersion relation without waveguide, $\omega = ck/\sqrt{\epsilon_r\,\mu_r}$ is hardly discernible.

It is often convenient to choose the dimensions of the waveguide so that at the operating frequency, only the lowest mode TE$_{1,0}$ can occur. Since the wave number, k_{mn}, is always less than the "free space" value, the wavelength in the waveguide is always larger than the free space wavelength. Surprisingly, this means that the phase velocity in free space is always greater than the free-space value, and equal to ω/k_{mn}.

The formulas given in Eq. (4.69) represent the transverse electric modes, where the z component of the electric field vanishes, and the electric field thus is "transverse" with respect to the propagation axis of the light waves in the wave guide.

Finally, let us give the formulas for the TM modes,

$$\boxed{\textbf{TM:}}\quad E_z(x,y) = E_{0,mn}\,\sin\left(\frac{m\pi x}{a}\right)\,\sin\left(\frac{n\pi y}{b}\right)\,\mathrm{e}^{\mathrm{i}\,(k\,z-\omega t)}\,,\qquad B_z(x,y)=0\,, \tag{4.74a}$$

$$\vec{E}_\parallel(m,n) = \mathrm{i}\frac{k}{\epsilon_r\,\mu_r\,\omega^2/c^2 - k^2}\,\vec{\nabla}_\parallel E_z = \mathrm{i}k\,\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{-1}\,E_{0,mn} \tag{4.74b}$$
$$\times\left[\hat{\mathrm{e}}_x\frac{m\pi}{a}\,\cos\left(\frac{m\pi x}{a}\right)\,\sin\left(\frac{n\pi y}{b}\right) + \hat{\mathrm{e}}_y\frac{n\pi}{b}\,\sin\left(\frac{m\pi x}{a}\right)\,\cos\left(\frac{n\pi y}{b}\right)\right]\mathrm{e}^{\mathrm{i}(k\,z-\omega t)}\,,$$

$$\vec{B}_\parallel(m,n) = \frac{\omega\,\epsilon_r\,\mu_r}{k\,c^2}\,\left(\hat{\mathrm{e}}_z\times\vec{E}_\parallel\right) = \mathrm{i}\,\frac{\epsilon_r\,\mu_r\,\omega}{c^2}\,\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{-1}\,E_{0,mn} \tag{4.74c}$$
$$\times\left[-\hat{\mathrm{e}}_x\frac{n\pi}{b}\,\sin\left(\frac{m\pi x}{a}\right)\,\cos\left(\frac{n\pi y}{b}\right) + \hat{\mathrm{e}}_y\frac{m\pi}{a}\,\cos\left(\frac{m\pi x}{a}\right)\,\sin\left(\frac{n\pi y}{b}\right)\right]\mathrm{e}^{\mathrm{i}(k\,z-\omega t)}\,.$$

The dispersion relation for TM modes is just the same as for TE modes, but the available values of m and n are different.

$$\omega^2 = \frac{c^2}{\epsilon_r\,\mu_r}\,\left(k^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right) = \frac{(ck)^2}{\epsilon_r\,\mu_r} + \omega_{mn}^2\,,\quad m=1,2,\dots,\quad n=1,2,\dots\,. \tag{4.75}$$

The z component of the electric field is $E_z(x,y)=0$ at $x=0$ and $x=a$, and at $y=0$ and $y=b$. In the TM modes, if $n=0$ or if $m=0$, then we have $E_z=0$, as a consequence of the properties of the sine versus the cosine function. So, the $m=1$, $n=0$ mode is not available for TM because the fields simply vanish. The lowest possible mode is $n=m=1$ with

$$\omega_{11} = \frac{\pi\,c}{\sqrt{\epsilon_r\mu_r}}\,\left[\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2\right]^{1/2} \tag{4.76}$$

$$\omega_{\mathrm{min;TM}} = \omega_{11} = \frac{\pi\,c}{\sqrt{\epsilon_r\mu_r}}\,\left[\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2\right]^{1/2} > \omega_{\mathrm{min;TE}} = \omega_{10} = \frac{\pi\,c}{a\sqrt{\epsilon_r\mu_r}}\,. \tag{4.77}$$

Thus, the transverse electric mode TE_{10} provides the smallest available frequency for propagation in the waveguide. For $a > b$, for frequencies

$$\omega_{\mathrm{min;TE}} < \omega < \omega_{\mathrm{min;TM}} \,, \tag{4.78}$$

only one mode is available for wave propagation. The phase velocity is given by:

$$\begin{aligned} v_p = \frac{\omega}{k} &= \frac{c}{\sqrt{\epsilon_r \mu_r}} \, \frac{\sqrt{k^2 + (\omega_{mn}/c)^2}}{k} = \frac{c}{\sqrt{\epsilon_r \mu_r}} \, \frac{\omega}{\sqrt{\omega^2 - \omega_{mn}^2}} \\ &= \frac{c}{\sqrt{\epsilon_r \mu_r}} \, \left(1 - \frac{\omega_{mn}^2}{\omega^2} \right)^{-1/2} > c \,. \end{aligned} \tag{4.79}$$

where the last inequality holds for $\epsilon_r = \mu_r = 1$ and $\omega_{mn} \neq 0$. The group velocity is given by:

$$\begin{aligned} v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} &= \frac{c}{\sqrt{\epsilon_r \mu_r}} \, \frac{\mathrm{d}}{\mathrm{d}k} [(ck)^2 + \omega_{mn}^2]^{1/2} = \frac{c^2 \, k}{\sqrt{\epsilon_r \mu_r}} \, \frac{1}{\sqrt{(ck)^2 + \omega_{mn}^2}} \\ &= \frac{c}{\sqrt{\epsilon_r \mu_r}} \, \frac{\sqrt{\omega^2 - \omega_{mn}^2}}{\omega} = \frac{c}{\sqrt{\epsilon_r \mu_r}} \, \left(1 - \frac{\omega_{mn}^2}{\omega^2} \right)^{1/2} \to 0 \,, \qquad \omega \to \omega_{mn} \,, \end{aligned} \tag{4.80}$$

where we indicate a few possible, alternative forms, and the latter limit indicates the possibility of "slow light" because the group velocity is the speed at which a light pulse (or, "light pulse train") travels. Furthermore, the functional form $\sqrt{\omega^2 - \omega_{mn}^2}/\omega$ suggests that the group velocity always remains smaller than the speed of light.

The index $n(\omega)$ of refraction can be determined from

$$k = n(\omega) \, \frac{\omega}{c} = \frac{\sqrt{\omega^2 - \omega_{mn}^2}}{c} \,, \tag{4.81a}$$

$$n(\omega) = \sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}} = \frac{1}{\sqrt{\epsilon_r \, \mu_r}} \, \frac{c}{v_p} < 1 \,, \tag{4.81b}$$

where the latter inequality holds for $v_p > c$.

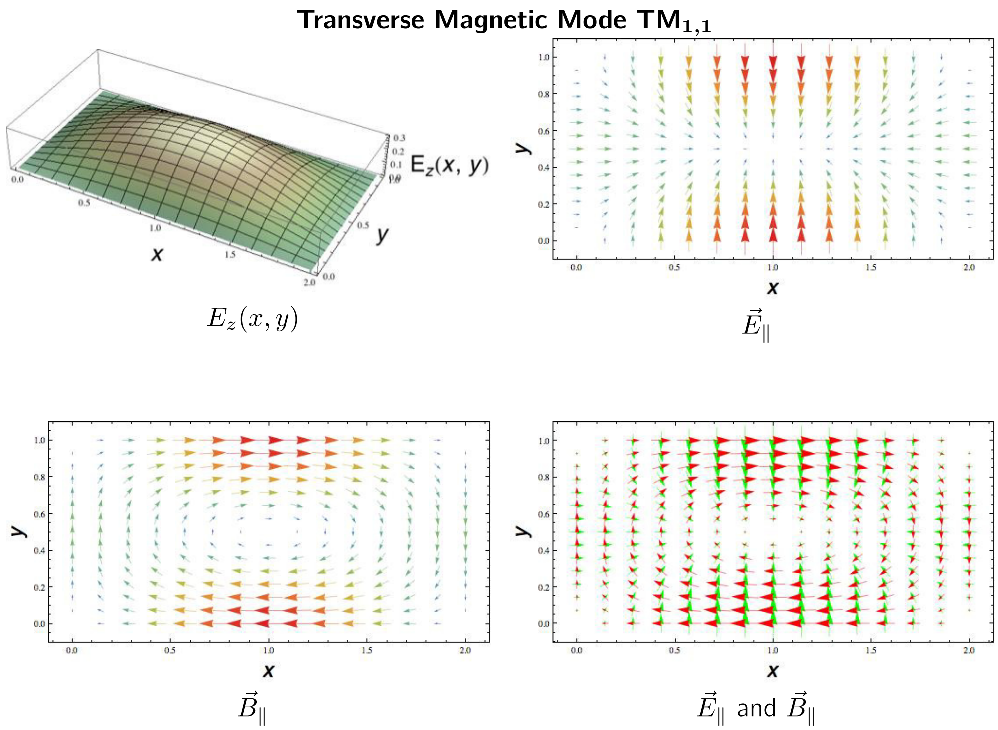


Figure 4.9: Four plots illustrating the transverse magnetic mode $\mathrm{TM}_{1,1}$. The leftmost upper plot shows $E_z(x,y)$ as a function of x and y. As the mode is transverse magnetic, $B_z(x,y)$ vanishes. The rightmost upper plot shows, as a vector plot, $\vec{E}_\parallel(x,y)$ in a plane of constant z, at a particular time of the sinusoidal oscillation. All vector arrows oscillate sinusoidally. The left lower plot shows $\vec{B}_\parallel(x,y)$ in a plane of constant z, at a particular time of the sinusoidal oscillation. The magnetic field lines form closed loops in the xy plane, as they do for all TM modes. Finally, in the right lower plot, the green arrows indicate the electric field, whereas the red ones indicate the magnetic field, illustrating the perpendicular character of the two fields.

4.3 Resonant Cavities

4.3.1 Resonant Cylindrical Cavities

Orientation. In a wave guide, modes are characterized by two discrete quantum numbers m and n, and a continuous quantum number k (or ω). The oscillations perpendicular to the z direction (propagation direction) are quantized. In a cylindrical cavity (as opposed to a wave guide), the oscillations in the third direction (z direction) are also quantized, and one ends up with a fully discrete set of modes. We start the discussion with the TM modes. The resonant modes are standing waves, not traveling waves as in the z-oriented waveguide, and resemble the “resonant modes” of, say, a drum or the string of a violin. In a wave guide, the allowed frequencies were of the form $\omega = \omega_{mn}(k)$, where m and n are discrete numbers (integers) but k is a continuous variable. The cavity adds one more boundary condition, in the z direction, because both the lower as well as the upper end are covered by a perfect conductor. This implies a further quantization condition, and we anticipate that the allowed cavity mode frequencies are of the form ω_{mnp},

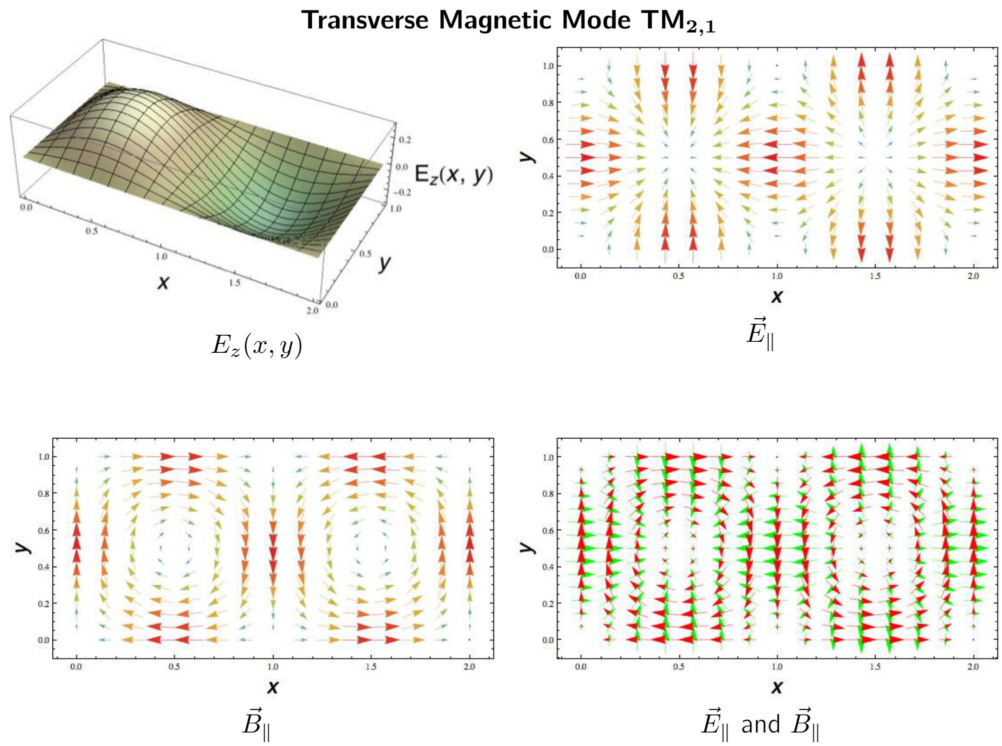


Figure 4.10: Four plots illustrating the transverse magnetic mode TM$_{2,1}$, otherwise the same same as Fig. 4.9.

with three integer subscripts.

TM Modes. We use cylindrical coordinates ρ, φ, and z. Let the cavity extend from $z = 0$ to $z = d$ and from $\rho = 0$ (symmetry axis) to $\rho = R$. We assume as before that

$$\vec{E}(\vec{r}, t) = \vec{E}(\vec{r})\, \mathrm{e}^{-\mathrm{i}\omega t}\,, \quad \vec{B}(\vec{r}, t) = \vec{B}(\vec{r})\, \mathrm{e}^{-\mathrm{i}\omega t}\,. \tag{4.82}$$

Let us assume first that the magnetic field is transverse, i.e., that

$$E_z = f(\rho)\, g(\varphi)\, h(z) = f(\rho)\, \mathrm{e}^{\mathrm{i}\, m\, \varphi} \left[C \, \sin\left(\frac{p\, \pi\, z}{d}\right) + D \, \cos\left(\frac{p\, \pi\, z}{d}\right) \right] . \tag{4.83}$$

The Helmholtz equation requires that

$$\begin{aligned} \left(\vec{\nabla}^2 + \epsilon_r \mu_r \frac{\omega^2}{c^2} \right) \, E_z &= \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \, \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} + \epsilon_r \mu_r \frac{\omega^2}{c^2} \right) \, E_z \\ &= \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} \underbrace{+ \frac{1}{\rho^2} \, \frac{\partial^2}{\partial \varphi^2}}_{= -m^2/\rho^2} + \underbrace{\left(- \left(\frac{p\, \pi}{d} \right)^2 + \epsilon_r \mu_r \, \frac{\omega^2}{c^2} \right)}_{= \gamma_p^2} \right] E_z \\ &= \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2} + \gamma_p^2 \right) \, E_z = 0 \, . \end{aligned} \tag{4.84}$$

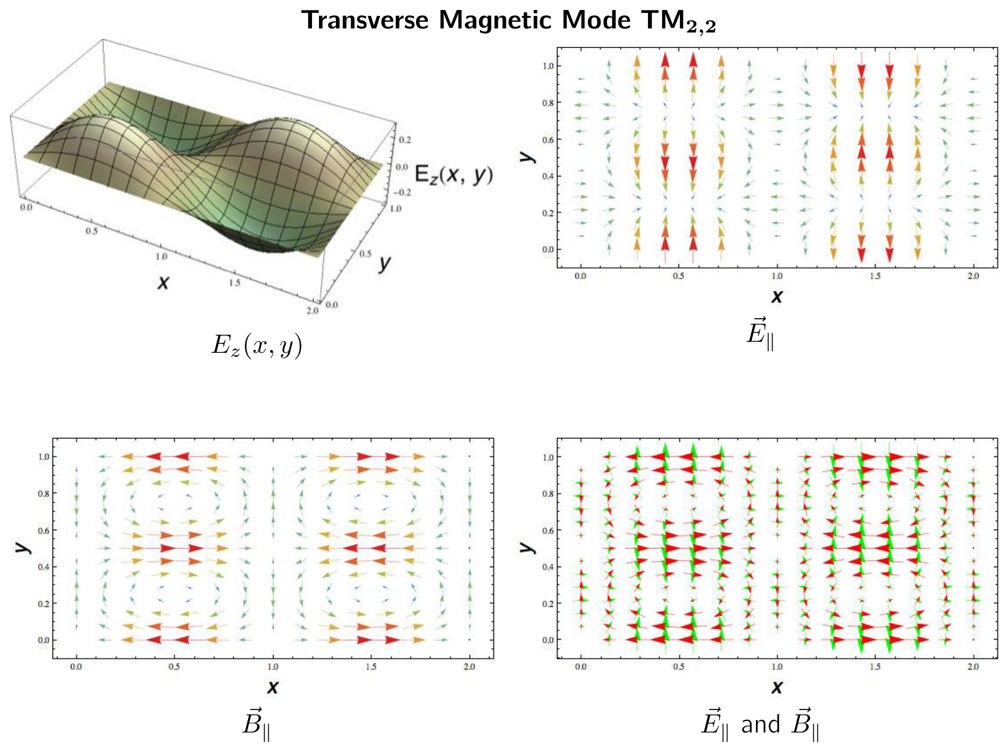


Figure 4.11: Four plots illustrating the transverse magnetic mode $\mathrm{TM}_{2,2}$, otherwise the same as Figs. 4.10 and 4.11.

The parameter γ_p is defined as

$$\gamma_p^2 = \epsilon_r \mu_r \frac{\omega^2}{c^2} - \left(\frac{p\,\pi}{d}\right)^2 . \tag{4.85}$$

We recall the defining differential equation for the ordinary Bessel function,

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{n^2}{\rho^2} + 1\right) J_n(\rho) = 0 . \tag{4.86}$$

The ansatz (4.83) can thus be specified as follows [$f(\rho) = E_0\, J_m(\gamma_p\, \rho)$],

Electric Field (z Component), Cylindrical Cavity, TM:

$$E_z = E_0\, J_m(\gamma_p\, \rho)\, \mathrm{e}^{\mathrm{i}\, m\, \varphi} \left[C\, \sin\left(\frac{p\,\pi\, z}{d}\right) + D\, \cos\left(\frac{p\,\pi\, z}{d}\right)\right] . \tag{4.87}$$

The constants C and D now have to be determined from the boundary conditions. We now concentrate on the TM modes with $B_z = 0$ everywhere. The gradient operator, in spherical coordinates, has the representation

$$\vec{\nabla} = \hat{\mathrm{e}}_\rho \frac{\partial}{\partial \rho} + \hat{\mathrm{e}}_\varphi \frac{1}{\rho}\frac{\partial}{\partial \varphi} + \hat{\mathrm{e}}_z \frac{\partial}{\partial z} = \vec{\nabla}_\parallel + \hat{\mathrm{e}}_z\, \nabla_z . \tag{4.88}$$

In general, because the z axis is still a valid symmetry axis of the problem, we can apply Eq. (4.19) to read as follows,

$$\nabla_z \vec{E}_\parallel + \mathrm{i}\omega \left(\hat{\mathrm{e}}_z \times \vec{B}_\parallel\right) = \vec{\nabla}_\parallel E_z . \tag{4.89}$$

Furthermore, we recall Eq. (4.23), which for $B_z = 0$ (transverse magnetic mode) reads as

$$\nabla_z \vec{B}_\parallel - \mathrm{i}\frac{\omega}{c^2}\,\epsilon_r\,\mu_r\,\left(\hat{\mathrm{e}}_z \times \vec{E}_\parallel\right) = \vec{\nabla}_\parallel B_z = 0\,. \tag{4.90}$$

In a waveguide, we could replace $\nabla_z \vec{E}_\parallel = \mathrm{i}k\vec{E}_\parallel$ and $\nabla_z \vec{B}_\parallel = \mathrm{i}k\vec{B}_\parallel$ without altering the $\mathrm{e}^{\mathrm{i}kz}$ dependence. We could thus solve Eq. (4.90) for $\vec{B}_\parallel$, after replacing $\nabla_z \vec{B}_\parallel \to \mathrm{i}\,k\,\vec{B}_\parallel$, and insert the result in Eq. (4.89), obtaining $\vec{E}_\parallel$ as a function of E_z. Now, $\vec{E}_\parallel$ and E_z will have different $\sin(p\,\pi\,z/d)$ and $\cos(p\,\pi\,z/d)$ dependences. and we have to integrate Eq. (4.90) with respect to z before using the relation in Eq. (4.89), or, in other words,

$$\nabla_z \vec{E}_\parallel = -\left(\frac{\pi\,p}{d}\right)^2 \int \vec{E}_\parallel\,\mathrm{d}z\,, \qquad \nabla_z^2 \vec{E}_\parallel = -\left(\frac{\pi\,p}{d}\right)^2 \vec{E}_\parallel\,, \qquad \vec{E}_\parallel = -\left(\frac{\pi\,p}{d}\right)^2 \int\!\!\int \vec{E}_\parallel\,\mathrm{d}z\,\mathrm{d}z\,. \tag{4.91}$$

A double integration with respect to $\mathrm{d}z$ thus is equivalent to a multiplication by $-(d/(\pi\,p))^2$. We can solve Eq. (4.90) for $\vec{B}_\parallel$ with the result

$$\vec{B}_\parallel = \mathrm{i}\,\frac{\omega}{c^2}\,\epsilon_r\,\mu_r\,\left(\hat{\mathrm{e}}_z \times \int \vec{E}_\parallel \mathrm{d}z\right) \tag{4.92}$$

and insert the result in the z-integrated form of Eq. (4.89),

$$\vec{E}_\parallel + \mathrm{i}\,\omega\left(\hat{\mathrm{e}}_z \times \int \vec{B}_\parallel \mathrm{d}z\right) = \int \vec{\nabla}_\parallel E_z \mathrm{d}z\,. \tag{4.93}$$

So, now, inserting (4.92) into (4.93), we have

$$\vec{E}_\parallel + \mathrm{i}\,\omega\left(\hat{\mathrm{e}}_z \times \mathrm{i}\,\frac{\omega}{c^2}\,\epsilon_r\,\mu_r\,\left(\hat{\mathrm{e}}_z \times \int\!\!\int \vec{E}_\parallel \mathrm{d}z\mathrm{d}z\right)\right) = \int \vec{\nabla}_\parallel E_z \mathrm{d}z\,. \tag{4.94}$$

We recall that, for any general $V_\parallel$, we have

$$\left(\hat{\mathrm{e}}_z \times \left(\hat{\mathrm{e}}_z \times \vec{V}_\parallel\right)\right) = -\vec{V}_\parallel\,. \tag{4.95}$$

So,

$$\vec{E}_\parallel - \mathrm{i}^2\,\frac{\omega^2}{c^2}\,\epsilon_r\,\mu_r \int\!\!\int \vec{E}_\parallel\,\mathrm{d}z\,\mathrm{d}z = \int \vec{\nabla}_\parallel E_z \mathrm{d}z\,, \tag{4.96}$$

or after a trivial transformation

$$\vec{E}_\parallel + \frac{\omega^2}{c^2}\,\epsilon_r\,\mu_r \int\!\!\int \vec{E}_\parallel\,\mathrm{d}z\,\mathrm{d}z = \int \vec{\nabla}_\parallel E_z\,\mathrm{d}z\,. \tag{4.97}$$

Within the ansatz (4.87), a double integration with respect to z is equivalent to a multiplication by $-(d/(p\pi))^2$, and so

$$\vec{E}_\parallel \left[1 - \frac{\omega^2}{c^2}\,\epsilon_r\,\mu_r\,\left(\frac{d}{p\pi}\right)^2\right] = \int \vec{\nabla}_\parallel E_z\,\mathrm{d}z\,. \tag{4.98}$$

We now pull out a prefactor,

$$\left(\frac{d}{p\pi}\right)^2 \vec{E}_\parallel \left[\left(\frac{p\pi}{d}\right)^2 - \frac{\omega^2}{c^2}\,\epsilon_r\,\mu_r\right] = \int \vec{\nabla}_\parallel E_z\,\mathrm{d}z\,, \tag{4.99}$$

so that

$$-\left(\frac{p\pi}{d}\right)^{-2} \gamma_p^2\,\vec{E}_\parallel = \int \vec{\nabla}_\parallel E_z\,\mathrm{d}z\,, \qquad \gamma_p^2 = \frac{\omega^2}{c^2}\,\epsilon_r\,\mu_r - \left(\frac{p\pi}{d}\right)^2\,. \tag{4.100}$$

In the ansatz given by Eq. (4.87), we choose the term with the cosine as opposed to the sine, for reasons which become obvious in the following, integrate with respect to z, and have

$$\vec{E}_\parallel = -\left(\frac{p\pi}{d}\right)^2 \left(\frac{d}{p\pi}\right) \frac{1}{\gamma_p^2} E_0 \, \vec{\nabla}_\parallel J_m(\gamma_p \, \rho) \, \sin\left(\frac{p\pi z}{d}\right) \, \mathrm{e}^{\mathrm{i}m\varphi} \, \mathrm{e}^{-\mathrm{i}\omega t} \,. \tag{4.101}$$

This formula can be summarized as

$$\vec{E}_\parallel = -E_0 \left(\frac{p\pi}{d}\right) \frac{1}{\gamma_p^2} \vec{\nabla}_\parallel J_m(\gamma_p \, \rho) \, \sin\left(\frac{p\pi z}{d}\right) \, \mathrm{e}^{\mathrm{i}m\varphi} \, \mathrm{e}^{-\mathrm{i}\omega t} \,. \tag{4.102}$$

The final result for $\vec{B}_\parallel$ is

$$\vec{B}_\parallel = \mathrm{i} \, \frac{\omega}{c^2} \, \epsilon_r \, \mu_r \left(\hat{\mathrm{e}}_z \times \int \vec{E}_\parallel \mathrm{d}z \right) \tag{4.103}$$

or

$$\vec{B}_\parallel = E_0 \left(\mathrm{i} \, \frac{\omega}{c^2} \, \epsilon_r \, \mu_r\right) \frac{1}{\gamma_p^2} \left(\hat{\mathrm{e}}_z \times \vec{\nabla}_\parallel J_m(\gamma_p \, \rho)\right) \cos\left(\frac{p\pi z}{d}\right) \, \mathrm{e}^{\mathrm{i}m\varphi} \, \mathrm{e}^{-\mathrm{i}\omega t} \,. \tag{4.104}$$

Here, the result

$$\int \sin\left(\frac{p\pi z}{d}\right) \mathrm{d}z = -\frac{d}{p\pi} \, \cos\left(\frac{p\pi z}{d}\right) \tag{4.105}$$

has been used. We summarize the result for TM modes,

Cylindrical Cavity, TM Modes:

$$B_z = 0 \,, \qquad E_z = E_0 \, J_m(\gamma_p \, \rho) \, \cos\left(\frac{p\pi z}{d}\right) \, \mathrm{e}^{\mathrm{i}m\varphi} \, \mathrm{e}^{-\mathrm{i}\omega t} \,, \tag{4.106a}$$

$$\vec{E}_\parallel = -\left(\frac{p\pi}{d}\right) \frac{1}{\gamma_p^2} E_0 \, \vec{\nabla}_\parallel J_m(\gamma_p \, \rho) \, \sin\left(\frac{p\pi z}{d}\right) \, \mathrm{e}^{\mathrm{i}m\varphi} \, \mathrm{e}^{-\mathrm{i}\omega t} \,, \tag{4.106b}$$

$$\vec{B}_\parallel = E_0 \left(\mathrm{i} \, \frac{\omega}{c^2} \, \epsilon_r \, \mu_r\right) \frac{1}{\gamma_p^2} \left(\hat{\mathrm{e}}_z \times \vec{\nabla}_\parallel J_m(\gamma_p \, \rho)\right) \cos\left(\frac{p\pi z}{d}\right) \, \mathrm{e}^{\mathrm{i}m\varphi} \, \mathrm{e}^{-\mathrm{i}\omega t} \,, \tag{4.106c}$$

$$\gamma_p^2 = \gamma_{mnp}^2 = \frac{\omega^2}{c^2} \, \epsilon_r \, \mu_r - \left(\frac{p\pi}{d}\right)^2 = \left(\frac{x_{mn}}{R}\right)^2 \,, \tag{4.106d}$$

$$\omega = \omega_{mnp} = \frac{c}{\sqrt{\epsilon_r \, \mu_r}} \sqrt{\left(\frac{p\pi}{d}\right)^2 + \left(\frac{x_{mn}}{R}\right)^2} \,, \qquad J_m(x_{mn}) = 0 \,. \tag{4.106e}$$

Here, x_{mn} is the nth zero of the Bessel function of order m. A posteriori, we can now understand why in Eq. (4.87), we could choose the term with the cosine only. Namely, any contribution of the sine term in Eq. (4.87) would otherwise lead to a term proportional to $\cos(p\,\pi\,z/d)$ in the expression for $\vec{E}_\parallel$. Now, in contrast to the term $\sin(p\,\pi\,z/d)$, a term proportional to $\cos(p\,\pi\,z/d)$ in $\vec{E}_\parallel$ does not vanish for $z = 0$ or $z = d$. However, the vector $\vec{E}_\parallel$ actually lies in the xy plane. So, it is parallel to the surface of the cylinder in the cylinder endcaps. By assumption, we have a perfect conductor material in the walls of the cylinder, so a parallel electric field component is not allowed.

The quantum numbers are m (azimuthal), n (radial) and p (longitudinal). The fundamental TM mode in a cylindrical cavity is the 010 mode ($m = 0$, $n = 1$, and $p = 0$). The numerical value is $x_{01} = 2.40482555769\ldots$. It has the lowest frequency available,

Cylindrical Cavity, Fundamental TM Mode:

$$\omega_{010} = \frac{c}{\sqrt{\epsilon_r \, \mu_r}} \, \frac{x_{01}}{R} \approx \frac{c}{\sqrt{\epsilon_r \, \mu_r}} \, \frac{2.404\,826}{R} \,. \tag{4.107}$$

This frequency is less than the corresponding minimum frequency for the lowest TE mode, and so the fundamental mode for a cylindrical cavity is TM rather than TE.

Indeed, not all values of γ_p are compatible with the boundary conditions, and we shall investigate them a little further, here. In view of the condition $E_z\big|_S = 0$, we recall that the z component of the electric field has to vanish on the outer surface of the cylindrical cavity, $E_z = 0$ at $\rho = R$. Let x_{mn} be the nth zero of the Bessel function of degree m,

$$J_m(x_{mn}) = 0\,, \qquad m = 0, 1, 2, \ldots, \qquad n = 1, 2, 3, \ldots\,. \tag{4.108}$$

Because of the condition that E_z must vanish on the outer rim of the cylinder, we must postulate that

$$\gamma_p \overset{!}{=} \gamma_{mn} = \gamma_{mnp} = \frac{x_{mn}}{R}\,, \qquad n = 1, 2, 3, \ldots \tag{4.109}$$

For each value of p, there is thus a resonant frequency,

$$\omega_{mnp} = \frac{c}{\sqrt{\epsilon_r \mu_r}} \left[\left(\frac{x_{mn}}{R}\right)^2 + \left(\frac{p\,\pi}{d}\right)^2\right]^{1/2}, \quad m = 0, 1, 2, \ldots\,, \quad n = 1, 2, 3, \ldots\,, \quad p = 0, 1, 2, \ldots\,. \tag{4.110}$$

The numbers m, n and p are the quantum numbers of the modes bound to the cavity. The quantum number n start from $n = 1$ because we must integrate at least up to the first node of the Bessel function, m starts from zero because the magnetic quantum number may vanish, and p also can be zero because E_z is proportional to $\cos(p\pi z/d)$, not $\sin(p\pi z/d)$, so the fields do not all vanish if we set $p = 0$. The quantum number n counts the nodes of the radial wave function and acts, in some sense, as the principal quantum number (the number of nodes actually is $n - 1$ for given n). The quantum number m acts as a projection quantum number characterizing the "angular momentum" about the quantization axis, $L_z = -\mathrm{i}\partial/\partial\varphi$. The quantum number p characterizes the number of oscillations in the longitudinal direction. The ground state is a radially symmetric state (no φ dependence) with quantum numbers $n = 1$, $m = 0$, and $p = 0$.

The TM states with $m = 0$ and $p = 0$ are characterized by the wave functions,

$$\boxed{\textbf{TM Mode } (m = p = 0)\textbf{:}} \quad E_z = E_0\, J_0(\gamma_{0n}\,\rho)\, \mathrm{e}^{-\mathrm{i}\omega_{0n0}\,t}\,, \qquad \vec{E}_\parallel = 0\,,$$

$$\vec{B}_\parallel = E_0\, \frac{\mathrm{i}\omega\epsilon_r\mu_r}{\gamma_{0n}^2\, c^2} \left(\hat{\mathrm{e}}_z \times \vec{\nabla}_\parallel J_0(\gamma_{0n}\,\rho)\right) \mathrm{e}^{-\mathrm{i}\omega_{0n0}\,t}\,, \qquad B_z = 0\,. \tag{4.111}$$

Of course, $J_0(\gamma_{0n}\,R) = 0$. The wave function with $n = 1$ has one nodes, with $n = 2$, two nodes, etc., much like in quantum mechanics. This is illustrated in Fig. 4.12. The electric and magnetic fields (at maximum amplitude) of the ground state $m = p = 0$ and $n = 1$ are shown in Fig. 4.13. In view of Eq. (4.111), the oscillations of magnetic and electric field are 90° out of phase for the standing wave.

Because of the longitudinal electric field, TM modes are good for accelerator cavities in storage rings. We suppress the arguments of the electric field in our notation, i.e., we understand that $E_z = E_z(\rho, \varphi, z, t)$, $\vec{E}_\parallel = \vec{E}_\parallel(\rho, \varphi, z, t)$, and $\vec{B}_\parallel = \vec{B}_\parallel(\rho, \varphi, z, t)$. The modes are not propagating, and they will be fully quantized. The in-plane component $\vec{E}_\parallel$ of the electric field vanishes alongside the outer rim of the cylinder, and also on the endcaps, $\vec{E}_\parallel|_S = \vec{0}$.

TE Modes. Let us now discuss the TE modes, where the electric field is transverse. Hence,

$$E_z = 0\,, \qquad B_z = B_0\, J_m(\overline{\gamma}_p\,\rho)\, \sin\left(\frac{p\pi z}{d}\right) \mathrm{e}^{\mathrm{i}m\varphi}\, \mathrm{e}^{-\mathrm{i}\omega t}\,. \tag{4.112}$$

The $\overline{\gamma}_p$ are assigned differently as compared to the γ_p for the TM modes, as we shall discuss in the following. Because the magnetic field has to be transverse at the outer boundary of the cylinder, we have to require that

$$\overline{\gamma}_p = \overline{\gamma}_{mnp} = \frac{\overline{x}_{mn}}{R}\,, \qquad J'_m(\overline{x}_{mn}) = 0\,. \tag{4.113}$$

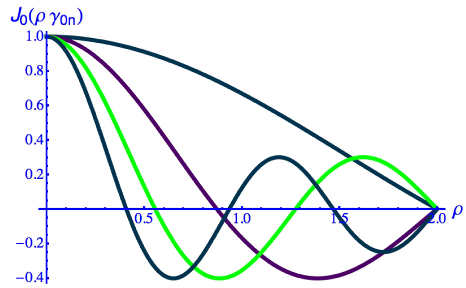


Figure 4.12: Illustration of the first few radial wave functions for the cylindrical cavity.

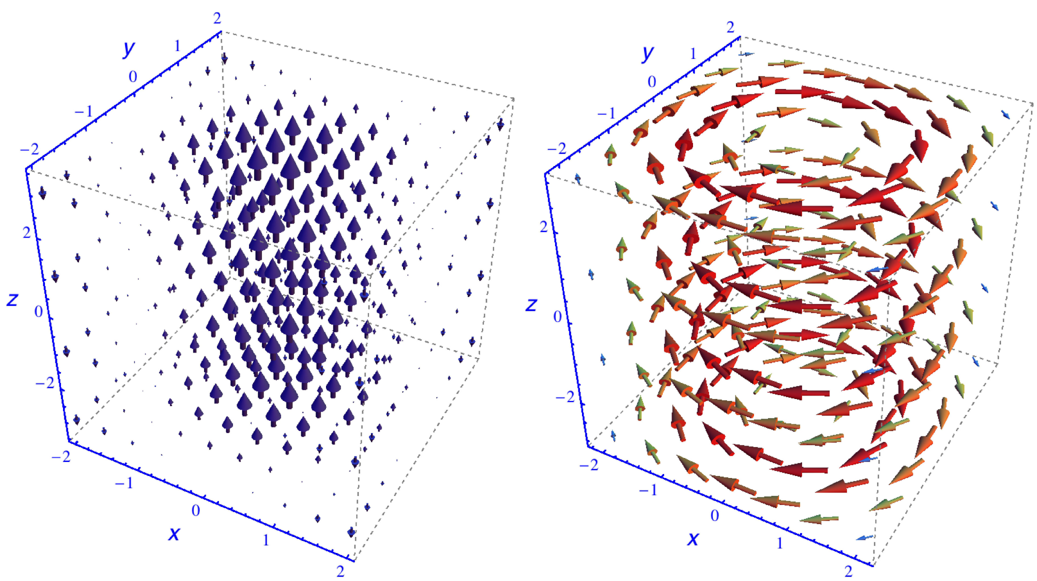


Figure 4.13: Illustration of the electric field for the TM ground state of the cylindrical cavity (left plot), and of the magnetic field (right plot). The electric field has a z component (and no transverse component), while the magnetic field is transverse and circulating around the symmetry axis.

For TE modes, we have in general,

$$\nabla_z \vec{E}_\parallel + \mathrm{i}\,\omega \left(\hat{\mathrm{e}}_z \times \vec{B}_\parallel \right) = \vec{\nabla}_\parallel E_z = \vec{0}\,, \tag{4.114}$$

$$\nabla_z \vec{B}_\parallel - \mathrm{i}\,\frac{\omega}{c^2}\,\epsilon_r\,\mu_r \left(\hat{\mathrm{e}}_z \times \vec{E}_\parallel \right) = \vec{\nabla}_\parallel B_z\,. \tag{4.115}$$

We integrate the latter equation with respect to z and obtain

$$\vec{B}_\parallel - \mathrm{i}\,\frac{\omega}{c^2}\,\epsilon_r\,\mu_r \left(\hat{\mathrm{e}}_z \times \int \vec{E}_\parallel \mathrm{d}z \right) = \int \vec{\nabla}_\parallel B_z \mathrm{d}z\,. \tag{4.116}$$

From the first equation, we have for $E_z = 0$,

$$\nabla_z \vec{E}_\parallel = -\mathrm{i}\,\omega \left(\hat{\mathrm{e}}_z \times \vec{B}_\parallel \right) \tag{4.117}$$

or, integrating,

$$\vec{E}_\parallel = -\mathrm{i}\,\omega \left(\hat{\mathrm{e}}_z \times \int \vec{B}_\parallel \mathrm{d}z \right) . \tag{4.118}$$

Inserting Eq. (4.118) into (4.116), we have the following relation,

$$\vec{B}_\parallel - \frac{\omega^2}{c^2}\,\epsilon_r\,\mu_r \left(\hat{\mathrm{e}}_z \times \left(\hat{\mathrm{e}}_z \times \int \vec{B}_\parallel \mathrm{d}z \right) \right) = \int \vec{\nabla}_\parallel B_z \mathrm{d}z\,. \tag{4.119}$$

In the space of perpendicular vectors, we have $\left(\hat{\mathrm{e}}_z \times \left(\hat{\mathrm{e}}_z \times \vec{V}_\parallel \right) \right) = \hat{\mathrm{e}}_z \left(\hat{\mathrm{e}}_z \cdot \vec{V}_\parallel \right) - \hat{\mathrm{e}}_z^2\,\vec{V}_\parallel = -\vec{V}_\parallel$ and so

$$\vec{B}_\parallel + \frac{\omega^2}{c^2}\,\epsilon_r\,\mu_r \int\int \vec{B}_\parallel \mathrm{d}z \mathrm{d}z = \int \vec{\nabla}_\parallel B_z \mathrm{d}z\,. \tag{4.120}$$

Within the ansatz (4.112), a double integration with respect to z is equivalent to a multiplication by $-(d/(p\pi))^2$, and so

$$\vec{B}_\parallel \left[1 - \frac{\omega^2}{c^2}\,\epsilon_r\,\mu_r \left(\frac{d}{p\pi} \right)^2 \right] = \int \vec{\nabla}_\parallel B_z \mathrm{d}z\,. \tag{4.121}$$

We can rewrite this as

$$\left(\frac{d}{p\pi} \right)^2 \vec{B}_\parallel \left[\left(\frac{p\pi}{d} \right)^2 - \frac{\omega^2}{c^2}\,\epsilon_r\,\mu_r \right] = \int \vec{\nabla}_\parallel B_z \mathrm{d}z\,. \tag{4.122}$$

Furthermore, since

$$\overline{\gamma}_p^2 = \frac{\omega^2}{c^2}\,\epsilon_r\,\mu_r - \left(\frac{p\pi}{d} \right)^2 \tag{4.123}$$

we have

$$-\left(\frac{p\pi}{d} \right)^{-2} \overline{\gamma}_p^2\,\vec{B}_\parallel = \int \vec{\nabla}_\parallel B_z \mathrm{d}z\,. \tag{4.124}$$

We can then trivially integrate the ansatz (4.112) with respect to z,

$$-\left(\frac{p\pi}{d} \right)^{-2} \overline{\gamma}_p^2\,\vec{B}_\parallel = -\left(\frac{p\pi}{d} \right) B_0\,\vec{\nabla}_\parallel J_m(\overline{\gamma}_p\,\rho)\,\cos\left(\frac{p\pi z}{d} \right)\,\mathrm{e}^{\mathrm{i}m\varphi}\,\mathrm{e}^{-\mathrm{i}\omega t}\,. \tag{4.125}$$

Solving for $\vec{B}_\parallel$, we obtain

$$\vec{B}_\parallel = B_0 \left(\frac{p\pi}{d} \right) \frac{1}{\overline{\gamma}_p^2}\,\vec{\nabla}_\parallel J_m(\overline{\gamma}_p\,\rho)\,\cos\left(\frac{p\pi z}{d} \right)\,\mathrm{e}^{\mathrm{i}m\varphi}\,\mathrm{e}^{-\mathrm{i}\omega t}\,. \tag{4.126}$$

The boundary condition is fulfilled if

$$\overline{\gamma}_p^2 = \frac{\omega^2}{c^2}\,\epsilon_r\,\mu_r - \left(\frac{p\pi}{d} \right)^2 = \left(\frac{\overline{x}_{mn}}{R} \right)^2 . \tag{4.127}$$

Solving for ω and setting $\omega = \omega_{nmp}$, we have

$$\omega_{nmp} = \frac{c}{\sqrt{\epsilon_r\,\mu_r}} \sqrt{\left(\frac{p\pi}{d} \right)^2 + \left(\frac{\overline{x}_{mn}}{R} \right)^2}\,. \tag{4.128}$$

Finally, we can determine $\vec{E}_\parallel$,

$$\nabla_z \vec{E}_\parallel = -\,\mathrm{i}\omega \left(\hat{\mathrm{e}}_z \times \vec{B}_\parallel \right) , \qquad \vec{E}_\parallel = -\mathrm{i}\omega \left(\hat{\mathrm{e}}_z \times \int \vec{B}_\parallel \mathrm{d}z \right) , \tag{4.129}$$

$$\vec{E}_\parallel = -\,\mathrm{i}\omega\,B_0\,\frac{1}{\overline{\gamma}_p^2} \left(\hat{\mathrm{e}}_z \times \vec{\nabla}_\parallel J_m(\overline{\gamma}_p\,\rho) \right) \sin\left(\frac{p\pi z}{d} \right)\,\mathrm{e}^{\mathrm{i}m\varphi}\,\mathrm{e}^{-\mathrm{i}\omega t}\,. \tag{4.130}$$

For $p = 0$, the z component of the magnetic field in Eq. (4.112) vanishes, i.e.,

$$B_z = B_0\, J_m(\overline{\gamma}_p\, \rho)\, \sin\left(\frac{p\pi z}{d}\right)\, \mathrm{e}^{\mathrm{i}m\varphi}\, \mathrm{e}^{-\mathrm{i}\omega t} = 0\,, \qquad (p = 0)\,. \tag{4.131}$$

Also,

$$\vec{E}_\parallel = -\mathrm{i}\omega\, B_0\, \frac{1}{\overline{\gamma}_p^2}\, \left(\hat{\mathrm{e}}_z \times \vec{\nabla}_\parallel J_m(\overline{\gamma}_p\, \rho)\right)\, \sin\left(\frac{p\pi z}{d}\right)\, \mathrm{e}^{\mathrm{i}m\varphi}\, \mathrm{e}^{-\mathrm{i}\omega t} = \vec{0}\,, \qquad (p = 0)\,. \tag{4.132}$$

Also,

$$\vec{B}_\parallel = B_0\, \left(\frac{p\pi}{d}\right)\, \frac{1}{\overline{\gamma}_p^2}\, \vec{\nabla}_\parallel J_m(\overline{\gamma}_p\, \rho)\, \cos\left(\frac{p\pi z}{d}\right)\, \mathrm{e}^{\mathrm{i}m\varphi}\, \mathrm{e}^{-\mathrm{i}\omega t} = \vec{0}\,, \qquad (p = 0)\,. \tag{4.133}$$

This excludes the mode with $p = 0$ as a viable, nontrivial field configuration, and implies that the mode with $p = 0$ is excluded for TE modes, but, as we shall see later, not for TM modes.

We summarize for TE modes,

Cylindrical Cavity, TE Modes:

$$E_z = 0\,, \qquad B_z = B_0\, J_m(\overline{\gamma}_p\, \rho)\, \sin\left(\frac{p\pi z}{d}\right)\, \mathrm{e}^{\mathrm{i}m\varphi}\, \mathrm{e}^{-\mathrm{i}\omega t}\,, \tag{4.134a}$$

$$\vec{E}_\parallel = -\,\mathrm{i}\omega\, B_0\, \frac{1}{\overline{\gamma}_p^2}\, \hat{\mathrm{e}}_z \times \vec{\nabla}_\parallel J_m(\overline{\gamma}_p\, \rho)\, \sin\left(\frac{p\pi z}{d}\right)\, \mathrm{e}^{\mathrm{i}m\varphi}\, \mathrm{e}^{-\mathrm{i}\omega t}\,, \tag{4.134b}$$

$$\vec{B}_\parallel = B_0\, \left(\frac{p\pi}{d}\right)\, \frac{1}{\overline{\gamma}_p^2}\, \vec{\nabla}_\parallel J_m(\overline{\gamma}_p\, \rho)\, \cos\left(\frac{p\pi z}{d}\right)\, \mathrm{e}^{\mathrm{i}m\varphi}\, \mathrm{e}^{-\mathrm{i}\omega t}\,, \tag{4.134c}$$

$$\overline{\gamma}_p^2 = \overline{\gamma}_{mnp}^2 = \frac{\omega^2}{c^2}\, \epsilon_r\, \mu_r - \left(\frac{p\pi}{d}\right)^2 = \left(\frac{\overline{x}_{mn}}{R}\right)^2\,, \tag{4.134d}$$

$$\omega = \overline{\omega}_{mnp} = \frac{c}{\sqrt{\epsilon_r\, \mu_r}}\, \sqrt{\left(\frac{p\pi}{d}\right)^2 + \left(\frac{\overline{x}_{mn}}{R}\right)^2}\,, \qquad J_m'(\overline{x}_{mn}) = 0\,. \tag{4.134e}$$

We recall that $\overline{x}_{mn}$ is the nth zero of the derivative of the Bessel function of order m.

Just like for TM modes, the quantum numbers are m (azimuthal), n (radial) and p (longitudinal). One might think that the fundamental TM mode in a cylindrical cavity is the 011 mode ($m = 0$, $n = 1$, and $p = 1$). The reasoning is as follows: The lowest-order Bessel function has $m = 0$, the first zero of its derivative is at $n = 1$, and we have to have at least $p = 1$ because the magnetic field B_z is otherwise vanishing. The numerical value is easly found as $\overline{x}_{01} = 3.831\,705\,970\ldots$. However, the smallest zero of the Bessel function of order $m = 1$ actually occurs before the first zero of $J_0(x)$, namely, at $\overline{x}_{11} = 1.841\,183\,781\ldots$. Hence, the corresponding angular frequency for the fundamental TE mode is

$$\overline{\omega}_{111} = \frac{c}{\sqrt{\epsilon_r\, \mu_r}}\, \sqrt{\left(\frac{\pi}{d}\right)^2 + \left(\frac{\overline{x}_{11}}{R}\right)^2} \approx \frac{c}{\sqrt{\epsilon_r\, \mu_r}}\, \sqrt{\left(\frac{\pi}{d}\right)^2 + \left(\frac{1.841\,183}{R}\right)^2}\,. \tag{4.135}$$

TE modes are useful for giving a transverse deflection to a beam in an accelerator, but are not much use for providing acceleration.

For $d \to \infty$, the TE_{10} mode again becomes the fundamental mode, and we obtain a waveguide.

4.3.2 Resonant Rectangular Cavities

In order to describe the resonant TE and TM eigenmodes of a rectangular cavity, it is advantageous to start from the vector potential, i.e., from an *ansatz* where

$$\vec{A}(\vec{r},t) = \vec{A}(\vec{r})\,\mathrm{e}^{-\mathrm{i}\omega t}\,, \qquad \vec{\nabla}\cdot\vec{A}(\vec{r}) = 0\,, \qquad \Phi(\vec{r},t) = 0\,, \tag{4.136}$$

so that

$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t)\,, \qquad \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{A}(\vec{r},t)\,. \tag{4.137}$$

The first step in this process is to derive a wave equation for the vector potential. We shall first derive a wave equation for the vector potential, and start from the Ampere–Maxwell law in the absence of source terms,

$$\vec{\nabla}\times\vec{H}(\vec{r},t) = \frac{\partial}{\partial t}\vec{D}(\vec{r},t)\,. \tag{4.138}$$

Suppressing the space-time coordinates [arguments of the fields $\vec{E} = \vec{E}(\vec{r},t)$ and $\vec{B} = \vec{B}(\vec{r},t)$], we have in a material with relative permittivity ϵ_r and relative permeability μ_r,

$$\vec{H} = \frac{1}{\mu_r\,\mu_0}\,\vec{B} = \frac{1}{\mu_r\,\mu_0}\,\vec{\nabla}\times\vec{A} \tag{4.139}$$

and

$$\vec{D} = \epsilon_r\,\epsilon_0\,\vec{E} = -\epsilon_r\,\epsilon_0\,\frac{\partial}{\partial t}\vec{A}(\vec{r},t)\,. \tag{4.140}$$

Inserting Eqs. (4.139) and (4.140) into (4.138), one obtains

$$\frac{1}{\mu_r\mu_0}\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = -\epsilon_r\,\epsilon_0\,\frac{\partial^2}{\partial t^2}\vec{A}\,, \tag{4.141}$$

which for $\vec{\nabla}\cdot\vec{A} = 0$ [see Eq. (4.136)] results in

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \vec{\nabla}\,\left(\vec{\nabla}\cdot\vec{A}\right) - \vec{\nabla}^2\vec{A} = -\vec{\nabla}^2\vec{A} = -\frac{\mu_r\epsilon_r}{c^2}\,\frac{\partial^2}{\partial t^2}\vec{A}\,. \tag{4.142}$$

We thus obtain the wave equation for the vector potential,

$$\left(\frac{\mu_r\epsilon_r}{c^2}\,\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\vec{A} = 0\,, \qquad \Rightarrow \qquad \left(\vec{\nabla}^2 + \frac{\mu_r\epsilon_r\,\omega^2}{c^2}\right)\vec{A} = 0\,, \tag{4.143}$$

We write the vector $\vec{k}$ as follows,

$$k_x = k\,\sin\theta\,\cos\varphi\,, \qquad k_y = k\,\sin\theta\,\sin\varphi\,, \qquad k_z = k\,\cos\theta\,, \tag{4.144a}$$

$$\vec{k} = k_x\,\hat{e}_x + k_y\,\hat{e}_y + k_z\,\hat{e}_z\,. \tag{4.144b}$$

For the later discussion, it is intructive to remember that the angles θ and φ belong to $\vec{k}$, not to the coordinate vector $\vec{r}$. We now define the two polarization vectors for TE and TM modes,

$$\hat{\epsilon}_{\vec{k},\mathrm{TE}} = \sin\varphi\,\hat{\mathrm{e}}_x - \cos\varphi\,\hat{\mathrm{e}}_y\,, \tag{4.145a}$$

$$\hat{\epsilon}_{\vec{k},\mathrm{TM}} = -\cos\theta\,\cos\varphi\,\hat{\mathrm{e}}_x - \cos\theta\sin\varphi\,\hat{\mathrm{e}}_y + \sin\theta\,\hat{\mathrm{e}}_z\,. \tag{4.145b}$$

$$\vec{k}\cdot\hat{\epsilon}_{\vec{k},\mathrm{TE}} = \vec{k}\cdot\hat{\epsilon}_{\vec{k},\mathrm{TM}} = 0 \qquad \hat{\epsilon}_{\vec{k},\mathrm{TM}}\times\hat{\epsilon}_{\vec{k},\mathrm{TE}} = \hat{k} = \vec{k}/|\vec{k}|\,. \tag{4.145c}$$

We write the vector potential for the mode functions as

$$\vec{A}_{\vec{k},\lambda}(\vec{r},t) = A_{\vec{k},\lambda,x}(\vec{r},t)\,\hat{\mathrm{e}}_x + A_{\vec{k},\lambda,y}(\vec{r},t)\,\hat{\mathrm{e}}_z + A_{\vec{k},\lambda,z}(\vec{r},t)\,\hat{\mathrm{e}}_z\,, \tag{4.146}$$

where λ can be TE or TM, and

$$A_{\vec{k},\lambda,x} = \mathcal{A}_0 \sqrt{\frac{8}{V}}\,\epsilon_{\vec{k},\lambda,x}\,\cos(k_x x)\,\sin(k_y y)\,\sin(k_z z)\,\mathrm{e}^{-\mathrm{i}\,\omega\,t}\,, \tag{4.147a}$$

$$A_{\vec{k},\lambda,y} = \mathcal{A}_0 \sqrt{\frac{8}{V}}\,\epsilon_{\vec{k},\lambda,y}\,\sin(k_x x)\,\cos(k_y y)\,\sin(k_z z)\,\mathrm{e}^{-\mathrm{i}\,\omega\,t}\,, \tag{4.147b}$$

$$A_{\vec{k},\lambda,z} = \mathcal{A}_0 \sqrt{\frac{8}{V}}\,\epsilon_{\vec{k},\lambda,z}\,\sin(k_x x)\,\sin(k_y y)\,\cos(k_z z)\,\mathrm{e}^{-\mathrm{i}\,\omega\,t}\,. \tag{4.147c}$$

Here, $V = L_x\,L_y\,L_z$, and $\mathcal{A}_0$ is a global amplitude normalized so that

$$\int_V \mathrm{d}^3 r\,\left|\vec{A}_{\vec{k},\lambda}(\vec{r},t)\right|^2 = \mathcal{A}_0^2\,. \tag{4.148}$$

The polarization vector for the mode with wave vector $\vec{k}$ and polarization λ is

$$\hat{\epsilon}_{\vec{k},\lambda} = \epsilon_{\vec{k},\lambda,x}\,\hat{\mathrm{e}}_x + \epsilon_{\vec{k},\lambda,y}\,\hat{\mathrm{e}}_y + \epsilon_{\vec{k},\lambda,z}\,\hat{\mathrm{e}}_z \tag{4.149}$$

and can describe a TE or TM mode, according to Eq. (4.145). The components of the wave vectors for the discrete modes are given as follows,

$$k_x = \frac{\ell\,\pi}{L_x}\,, \qquad k_y = \frac{m\,\pi}{L_y}\,, \qquad k_z = \frac{n\,\pi}{L_z}\,, \qquad \vec{k}_{\ell mn} = \frac{\ell\,\pi}{L_x}\,\hat{\mathrm{e}}_x + \frac{m\,\pi}{L_y}\,\hat{\mathrm{e}}_y + \frac{n\,\pi}{L_z}\,\hat{\mathrm{e}}_z\,. \tag{4.150}$$

where ℓ, m, n are integer. The electric field corresponding to the polarization $\lambda = \mathrm{TE}$ is transverse, i.e., its z component vanishes, while the magnetic induction field corresponding to the polarization $\lambda = \mathrm{TM}$ is transverse, i.e., its z component vanishes. The boundary conditions for the electric field and the boundaries of the rectangular cavity are fulfilled if Eq. (4.150) is fulfilled.

The transversality condition (4.136) for the vector potential is verified as follows,

$$\vec{\nabla}\cdot\vec{A}_{\vec{k},\lambda}(\vec{r}) = -\mathcal{A}_0 \sqrt{\frac{8}{V}}\,\left(\vec{k}\cdot\hat{\epsilon}_{\vec{k},\lambda}\right)\,\sin(k_x x)\,\sin(k_y y)\,\sin(k_z z) = 0\,, \tag{4.151}$$

which is fulfilled if $\vec{k}\cdot\vec{\epsilon}_{\vec{k}\lambda} = 0$. The wave equation (4.143), applied to the vector potential (4.147), results in the relation

$$k_x^2 + k_y^2 + k_z^2 = \frac{\epsilon_r\,\mu_r\omega^2}{c^2}\,. \tag{4.152}$$

Together with Eq. (4.150), this leads to the "quantization condition"

$$\omega = \omega_{\ell mn} = \frac{c}{\sqrt{\epsilon_r\,\mu_r}}\,\sqrt{\left(\frac{\ell\,\pi}{L_x}\right)^2 + \left(\frac{m\,\pi}{L_y}\right)^2 + \left(\frac{n\,\pi}{L_z}\right)^2}\,,$$

$$\ell = 0,1,2,\ldots, \qquad m = 0,1,2,\ldots, \qquad n = 0,1,2,\ldots\,. \tag{4.153}$$

In the rectangular wave guide, we have no continuous symmetry axis. Still, three quantum numbers and the polarization λ suffice to characterize all possible modes.

Let us try to identify the fundamental mode. In order for the vector potential in Eq. (4.147) to be nonvanishing, two of the "quantum numbers" ℓ, m, and n must be nonvanishing. Otherwise, the entire vector potential in Eq. (4.147) vanishes. First, let L_y and L_z be the longest edges of the rectangular cavity. Then, the angular frequency of the fundamental mode is

$$\omega_{011} = \frac{c}{\sqrt{\epsilon_r\,\mu_r}} \sqrt{\left(\frac{\pi}{L_y}\right)^2 + \left(\frac{\pi}{L_z}\right)^2}\,. \tag{4.154}$$

Expressed differently, the fundamental mode carries the quantum numbers $\ell = 0$, $m = 1$ and $n = 1$, with frequency ω_{011} given in Eq. (4.154). For the fundamental mode, we thus have $k_x = \ell\,\pi/L_x = 0$ and hence $\varphi = 90° = \pi/2$. According to Eq. (4.145), the two polarization vectors are thus given by

$$\hat{\epsilon}_{\vec{k}_{011},\mathrm{TE}} = \hat{\mathrm{e}}_x\,, \qquad \hat{\epsilon}_{\vec{k}_{011},\mathrm{TM}} = -\cos\theta\,\hat{e}_y + \sin\theta\,\hat{e}_z\,, \tag{4.155a}$$

$$\cos\theta = \frac{k_z}{\sqrt{k_y^2 + k_z^2}} = \frac{1/L_z}{\sqrt{(1/L_y)^2 + (1/L_z)^2}}\,. \tag{4.155b}$$

For $\ell = 0$, $m = 1$ and $n = 1$, the only nonvanishing component of $\vec{A}$ is the x component $A_{\vec{k}_{011},\mathrm{TE},x} \neq 0$ [see Eq. (4.147)]. One may combine Eqs. (4.150) with Eq. (4.147) and observe that $\sin(k_x\,x) = 0$. Furthermore, because only $\hat{\epsilon}_{\vec{k}_{011},\mathrm{TE}}$ has a nonvanishing x component, the fundamental mode is a TE mode if the longest edges are L_y and L_z.

Second, let L_x and L_y be the longest edges of the rectangular cavity. In this case, the angular frequency of the fundamental mode is

$$\omega_{110} = \frac{c}{\sqrt{\epsilon_r\,\mu_r}} \sqrt{\left(\frac{\pi}{L_x}\right)^2 + \left(\frac{\pi}{L_y}\right)^2}\,. \tag{4.156}$$

In this case, $\theta = 90°$. According to Eq. (4.145), the two relevant polarization vectors are

$$\hat{\epsilon}_{\vec{k}_{110},\mathrm{TE}} = \sin\varphi\,\hat{\mathrm{e}}_x - \cos\varphi\,\hat{\mathrm{e}}_y\,, \qquad \hat{\epsilon}_{\vec{k}_{110},\mathrm{TM}} = \hat{\mathrm{e}}_z\,, \tag{4.157a}$$

$$\cos\varphi = \frac{k_x}{\sqrt{k_x^2 + k_y^2}} = \frac{1/L_x}{\sqrt{(1/L_x)^2 + (1/L_y)^2}}\,. \tag{4.157b}$$

For $\ell = 1$, $m = 1$ and $n = 0$, the only nonvanishing component of $\vec{A}$ is the z component $A_{\vec{k}_{110},\mathrm{TE},z} \neq 0$. Furthermore, because only $\hat{\epsilon}_{\vec{k}_{110},\mathrm{TM}}$ has a nonvanishing z component, the fundamental mode is a TM mode if the shortest edge is L_z.

Let us try to verify once more, the explicit fulfillment of the boundary conditions by the electric and magnetic fields generated by the vector potentials indicated in Eq. (4.147). We recall that, in view of Eq. (4.137), we have $\vec{E} = -\partial\vec{A}/\partial t = \mathrm{i}\omega\vec{A}$. The transversal (normal) component of the electric field does not need to vanish at the boundaries. Indeed we have, e.g., for the planes with $z = 0$ or at $z = L_z$,

$$E_{\vec{k},\lambda,z}(x, y, L = 0) = E_{\vec{k},\lambda,z}(x, y, L = L_z) = -\mathrm{i}\omega\,\mathcal{A}_0\,\sqrt{\frac{8}{V}}\,\epsilon_{\vec{k},\lambda,z}\,\sin(k_x\,x)\,\sin(k_y\,y)\,. \tag{4.158}$$

However, the mode functions (4.147) are such that there is a cosine function if the Cartesian component of the wave vector in the argument of the trigonometric function is the same Cartesian component as the vector potential component itself. The presence of the sine functions implies that the tangential component of the vector potential (the component lying inside the boundary planes) vanishes for all three boundaries,

$$A_{\vec{k},\lambda,x}(x, y, z = 0) = A_{\vec{k},\lambda,x}(x, y, z = L_z) = A_{\vec{k},\lambda,x}(x, y = 0, z) = A_{\vec{k},\lambda,x}(x, y = L, z) = 0\,, \tag{4.159a}$$

$$A_{\vec{k},\lambda,y}(x, y, z = 0) = A_{\vec{k},\lambda,y}(x, y, z = L_z) = A_{\vec{k},\lambda,y}(x = 0, y, z) = A_{\vec{k},\lambda,y}(x = L, y, z) = 0\,, \tag{4.159b}$$

$$A_{\vec{k},\lambda,z}(x = 0, y, z) = A_{\vec{k},\lambda,z}(x = L, y, z) = A_{\vec{k},\lambda,z}(x, y = L, z) = A_{\vec{k},\lambda,z}(x, y = L, z) = 0\,. \tag{4.159c}$$

In view of $\vec{E} = -\partial \vec{A}/\partial t = \mathrm{i}\omega \vec{A}$, the conditions (4.159) are exactly equivalent to the boundary condition for the electric field, given in Eq. (4.44), namely, $\vec{E}_{\parallel,S} = 0$.

The next step is to look at the $\vec{B}$ field. An explicit calculation of the curl of the vector potential given in Eq. (4.147) leads to the following result for $\vec{B} = \vec{\nabla} \times \vec{A}$,

$$B_{\vec{k},\lambda,x} = \mathcal{A}_0 \sqrt{\frac{8}{V}} \left(\epsilon_{\vec{k},\lambda,z}\, k_y - \epsilon_{\vec{k},\lambda,y}\, k_z \right) \sin(k_x x)\, \cos(k_y y)\, \cos(k_z z)\, \mathrm{e}^{-\mathrm{i}\,\omega\, t} \,, \tag{4.160a}$$

$$B_{\vec{k},\lambda,y} = \mathcal{A}_0 \sqrt{\frac{8}{V}} \left(\epsilon_{\vec{k},\lambda,x}\, k_z - \epsilon_{\vec{k},\lambda,z}\, k_x \right) \cos(k_x x)\, \sin(k_y y)\, \cos(k_z z)\, \mathrm{e}^{-\mathrm{i}\,\omega\, t} \,, \tag{4.160b}$$

$$B_{\vec{k},\lambda,z} = \mathcal{A}_0 \sqrt{\frac{8}{V}} \left(\epsilon_{\vec{k},\lambda,y}\, k_x - \epsilon_{\vec{k},\lambda,x}\, k_y \right) \cos(k_x x)\, \cos(k_y y)\, \sin(k_z z)\, \mathrm{e}^{-\mathrm{i}\,\omega\, t} \,. \tag{4.160c}$$

Here, the sine term is with the same coordinate as the component of the $\vec{B}$ field; it means that the component in the direction of the normal to the boundary surface vanishes on the boundaries, fulfilling Eq. (4.46).

In our considerations leading up to Eqs. (4.155) and (4.157), we considered the fundamental mode for the cases of the shortest edge length being in the x and z directions. We found that we have a special situation where one of the mode functions given in Eq. (4.147) vanishes, and we only have one polarization available. In this case, one of the quantum numbers ℓ, m, n is zero. Let us try to generalize the consideration somewhat. If either ℓ, m, n is zero, then according to Eqs. (4.150) and (4.147), only one of the components of the vector potential $\vec{A}$ "survives"; the others vanish. The only valid polarization vector points into the same Cartesian direction as the vanishing component of $\vec{k}$. If $\ell = 0$, then we need to have $\hat{\epsilon}_{\vec{k},\lambda} = \hat{\mathrm{e}}_x$, and if $n = 0$, then $\hat{\epsilon}_{\vec{k},\lambda} = \hat{\mathrm{e}}_z$. This can be verified on the examples given in Eqs. (4.155) and (4.157) and generalizes to any mode with one vanishing quantum number.

4.4 Casimir Effect and Quantum Electrodynamics

Ships in the ocean travelling at constant speed, and close to each other, are known to attract each other because the waves are broken in between the ships; or in other words, the wave motion is attenuated between the ships. By contrast, no such attenuation occurs on the respective other side sides of the ships, namely, on the sides not opposing each other. Therefore, the waves exert forces on the ships, attenuated from in between but not attenuated from outside, and the two ships actually attract each other. Seamen know this effect, wherefore ships on the open sea have to keep a minimum distance from each other; and that includes ships traveling in convoys. The Casimir attraction of perfectly conducting plates is of the same origin.

Here, a brief overview is given of the effect, with a very brief glance at the field quantization eventually necessary in order to describe the effect. In our studies on the calculation of formally diverging potentials in PHYSICS/5211, we had already seen that the fundamental equation of quantum electrodynamics is:

$$\begin{aligned} \mathrm{Physical\_Observable} \;&=\; \mathrm{Bare\_Observable}\big|_{\mathrm{reg.}} \;-\; \mathrm{Counter\_Term}\big|_{\mathrm{reg.}} \\ &= \lim_{\mathrm{regulator}\to 0} \int \left(\mathrm{Bare\_Observable}\big|_{\mathrm{reg.-int.}} \;-\; \mathrm{Counter\_Term}\big|_{\mathrm{reg.-int.}} \right) . \end{aligned} \tag{4.161}$$

The meaning of these expressions is as follows:

- $\mathrm{Bare\_Observable}$: an expression describing the physical observable, as derived from first principles, invoking the formalism of quantized electromagnetic and fermion fields (i.e., loosely speaking, from the "normal Feynman rules of QED").

- $\text{Bare\_Observable}\big|_{\text{reg.}}$: a regularized version of the expression for the bare observable, where the regulator respects basic symmetries of the physical problem, as discussed in much greater detail below.
- Counter_Term: a counter-term is a quantity which needs to be absorbed into a parameter of the theory, i.e. interpreted as a physically unobservable contribution to an energy, or to a scattering amplitude, which needs to be subtracted in order to obtain a physically sensible answer.
- $\text{Counter\_Term}\big|_{\text{reg.}}$: a regularized version of the expression for the counter-term, where the regulator needs to be chosen so that it is compatible with the one used for the bare observable.
- $\text{Bare\_Observable}\big|_{\text{reg.-int.}}$: The regularization proceeds most commonly on the level of some integral, and so, without specifying the variables which need to be integrated over, we denote the regularized integrand by the symbol "reg.-int.".
- $\text{Counter\_Term}\big|_{\text{reg.-int.}}$: The regularized integrand for the counter-term.
- $\lim_{\text{regulator}\to 0}$: The regularization is removed after all other operations (including integrations) have been carried out, and only this sequence ensures that no spurious finite contributions remain in the final expressions, which could otherwise lead to inconsistent results.

The basic, fundamental Eq. (4.161) is illustrated in the current section. The physical situation is illustrated in Fig. 4.14. Two perfectly conducting plates are a distance R apart, and the space in between them is empty. However, there may still be a nonvanishing interaction, i.e. a nonvanishing (attractive) force between the plates, because the vacuum (the "nothing") is influenced by the presence of the plates. In other words, the discretization of the electromagnetic field modes, as implied by the presence of the plates, generates a small residual effect which leads to an attractive interaction of the plates. All aspects of the fundamental, conceptual Eq. (4.161) are thus important.

- Yes, we need a counterterm. Reason: the effect of the plates is only felt as a residual effect, as a change of the zero-point energy in view of the presence of the plates, as opposed to a situation where the plates are absent. However, the zero-point energy in the absence of the plates cannot be physically observable and must thus be subtracted. The approach thus consists of calculating the sum of the zero-point energies of the discretized modes in the presence of the plates, minus the sum of the zero-point energies of all the modes in the absence of the plates. The total zero-point energy in the presence of the plates should thus be interpreted as the unrenormalized quantity to be calculated, whereas the total zero-point energy in the absence of the plates is the counterterm.
- Yes, we need a regulator. The reason is that both the sum of the zero-point energies of the electromagnetic modes in the presence as well as in the absence of the plates are highly divergent quantities. From physical considerations, only the infrared region of long wavelengths is influenced by the quantization. The ultraviolet region does not receive any corrections. The divergence of both the unrenormalized quantity as well as of the counterterm can thus be remedied in a physically sensible way by the introduction of an ultraviolet regulator, i.e. by the introduction of a quantity which suppresses the divergence induced by the high-frequency electromagnetic modes. This ultraviolet regulator should respect basic symmetries of the problem. In the current case, it means that this ultraviolet regulator should respect, e.g., the rotational symmetry of the total arrangement in the limit $L \to \infty$.
- Yes, we should be careful about the order of integrations. The regulator (let us call it η), which induces an exponential damping of the contribution from the ultraviolet modes, should be kept up until the very end of the calculation. We should not perform the limit $\eta \to 0$ before all other integrations are done.

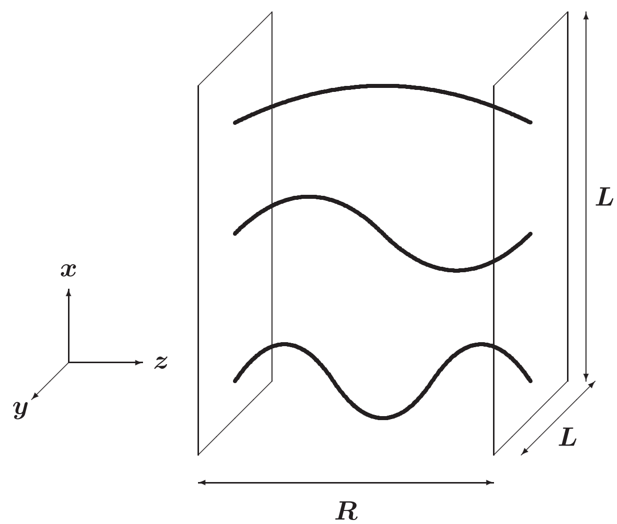


Figure 4.14: Casimir box. $L \gg R$. Some modes of the electromagnetic field (standing waves with vanishing electric-field amplitude on the plates) are sketched.

Our consideration starts with the calculation of a formal expression for the sum of zero-point energies of the electromagnetic modes between the plates. We shall interpret every mode of the electromagnetic field as a harmonic oscillator, which is n-fold excited when there are n photons present in the mode, and we simply take this concept for granted in the current derivation, while we stress that a rigorous justification will follow later. We should also point out that the derivation as given below relies on the *quantization* of the electromagnetic field.

We consider the zero-point energy of a harmonic oscillator of frequency ω. The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \,. \tag{4.162}$$

Annihilation and creation operators may be defined as follows,

$$a = \sqrt{\frac{1}{2}} \left(\frac{x}{x_0} + \mathrm{i}\, \frac{x_0\, p}{\hbar} \right) , \qquad a^+ = \sqrt{\frac{1}{2}} \left(\frac{x}{x_0} - \mathrm{i}\, \frac{x_0\, p}{\hbar} \right) . \tag{4.163}$$

The length scale x_0 is given by

$$x_0 = \sqrt{\frac{\hbar}{m\omega}} \,. \tag{4.164}$$

The Hamiltonian may thus be rewritten as

$$H = \hbar\omega \left(a^+\, a + \frac{1}{2} \right) , \tag{4.165}$$

where $a^+\, a$ is a number operator that counts the number of oscillation *quanta* (photons). Its spectrum consists of the nonnegative integers. The zero-point energy (of the fundamental mode) can thus be inferred as

$$e_0 = \tfrac{1}{2}\, \hbar\omega \,. \tag{4.166}$$

Normally, this zero-point energy has no physically observable effect. However, as shown by Casimir and Polder, a small change in the total zero-point energies of a collection of oscillators (in this case, modes of the electromagnetic field), between two parallel perfect mirrors, has a nonvanishing small effect which results in an attractive interaction between the plates.

We consider two square perfectly conducting metal plates (of dimension $L \times L$, see Fig. 4.14), which define planes parallel to the xy-plane, a distance R apart. The plates are supposed to act as perfect mirrors over the entire frequency range of possible photon modes, which is of course an idealization. Furthermore, we assume that the plate dimension L is much larger than the plate distance, $L \gg R$.

The photon modes in between the plates are subject to the condition that the electric field should vanish on the plates. For standing waves, the condition is that R, the distance between the plates, must be an integer multiple of half wavelengths, i.e.

$$R = n_z \left(\frac{1}{2}\right) \left(\frac{2\pi}{k_z}\right) , \qquad n_z = 0, 1, 2, \dots . \tag{4.167}$$

This discretizes the available modes (standing waves) in the z-direction, which can be associated to a quantum number n_z,

$$k_z = n_z \, \frac{\pi}{R} , \qquad n_z = 0, 1, 2, \dots . \tag{4.168}$$

For the x and y components of the photon momentum vector, no such discretization is indicated, and we can assume traveling (not standing!) waves which are associated with non-discretized quantum numbers n_x and n_y,

$$k_x = n_x \, \frac{2\pi}{L} , \qquad k_y = n_y \, \frac{2\pi}{L} , \qquad n_x, \, n_y \in \mathbb{R} . \tag{4.169}$$

Here, n_x and n_y can acquire positive as well as negative values. Summations over n_x and n_y can therefore be replaced by integrations,

$$\sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} \to \left(\frac{L}{2\pi}\right)^2 \int_{-\infty}^{\infty} \mathrm{d}k_x \int_{-\infty}^{\infty} \mathrm{d}k_y = L^2 \int_{-\infty}^{\infty} \frac{\mathrm{d}k_x}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_y}{2\pi} . \tag{4.170}$$

We might just as well have assumed standing waves in counting the k_x and k_y, but then the integration limits would have been 0 and ∞ (standing waves do not have a propagation direction), and the prefactor would change to $(L/\pi)^2$, leading to identical final results. The relations (4.169) correspond to the usual counting of available vacuum modes that is being used in many derivations in physics, e.g., in the derivation of the Planck law for blackbody radiation.

According to Eq. (4.166), the zero-point energy of a harmonic oscillator of frequency ω is $e_0 = \frac{1}{2}\hbar\omega$. We may thus calculate the sum of the zero-point energies of the photon modes in the space confined by the two plates as follows, keeping track of the summation limits and the photon polarizations λ carefully (here, $\lambda_{\max}$ characterizes the maximum number of polarization modes available for a given electromagnetic mode with wave vector $\vec{k}$),

$$\begin{aligned} E_{\mathrm{bare}} = & \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} \sum_{n_z=0}^{\infty} \sum_{\lambda=1}^{\lambda_{\max}} e_0 = \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} \sum_{n_z=0}^{\infty} \sum_{\lambda=1}^{\lambda_{\max}} \frac{1}{2} \, \hbar \, c \, |\vec{k}_{n_x,n_y,n_z}| \\ = & \frac{\hbar \, c}{2} \, L^2 \int_{-\infty}^{\infty} \frac{\mathrm{d}k_x}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_y}{2\pi} \sum_{n_z=0}^{\infty} \sum_{\lambda=1}^{\lambda_{\max}} \sqrt{(k_x)^2 + (k_y)^2 + \frac{(n_z)^2 \pi^2}{R^2}} \\ = & \frac{\hbar \, c \, L^2}{8\pi^2} \int_{\mathbb{R}^2} \mathrm{d}^2 k_\parallel \sum_{n_z=0}^{\infty} \sum_{\lambda=1}^{\lambda_{\max}} \sqrt{\vec{k}_\parallel^2 + \frac{(n_z)^2 \, \pi^2}{R^2}} . \end{aligned} \tag{4.171}$$

The specification "bare" means that it refers to an unrenormalized, unregularized quantity which may be divergent (and actually is divergent in our case). The notation $\mathrm{d}^2 k_\parallel$ is chosen for the integral $\mathrm{d}k_x\,\mathrm{d}k_y$.

The sum over polarizations λ can now be evaluated as follows: for the mode with $n_z = 0$, we would assume, in principle, the availability of two possible polarizations for the photon as it propagates in the xy-plane, i.e., parallel to the plates and in the space "in between." Let us assume that the wave propagates in the x direction. Then, the electric field of the wave may be directed in the z or the y direction. However, a strictly y polarized wave would have its electric field pointing parallel to the plates. This would imply a a nonvanishing component of the photon electric field oscillating, in particular, in the xy-plane, in contradiction to our assumption of perfectly conducting plates. So, $\lambda_{\max} = 1$ for $n_z = 0$.

For the modes with $n_z > 0$, we have two polarizations available: the standing waves offer two possible polarizations (the electric field vector may be aligned along the x or y axes). By contrast, the electric field that characterizes the motion of the photon in the xy-plane is again restricted to one orientation. We have $n_{\max} = 2$ for $n_z > 0$. The different number of allowed polarizations finds a natural mathematical complement in terms of the Euler–Maclaurin summation formula, as discussed below. We obtain

$$E_{\rm bare} = \frac{\hbar\, c\, L^2}{8\pi^2} \int_{\mathbb{R}^2} \mathrm{d}^2 k_\parallel \left(\left| \vec{k}_\parallel \right| + 2 \sum_{n_z=1}^{\infty} \sqrt{\vec{k}_\parallel^2 + \frac{n_z^2 \pi^2}{R^2}} \right) . \tag{4.172}$$

One might have expected from the facts that (i) the zero-point energy of any particular harmonic oscillator mode is finite, and (ii) there is an infinite number of such modes, that the total zero-point energy of the entire system should be a highly divergent quantity. This is confirmed by Eq. (4.172). Indeed, $E_{\rm bare}$ diverges as

$$\int_{|\vec{k}_\parallel|<\Lambda} \mathrm{d}^2 k_\parallel \, |\vec{k}_\parallel| \sim \int_0^\Lambda \mathrm{d}k_\parallel \, |\vec{k}_\parallel|^2 \sim \Lambda^3 \tag{4.173}$$

and therefore requires a strong regulator for large cutoff Λ. We must thus transform

$$E_{\rm bare} \to E_{\rm bare}\big|_{\rm reg.} \tag{4.174}$$

in the sense of Eq. (4.161), by the introduction of a suitable regularization.

What should we do now? The answer is provided by the concepts of regularization and renormalization. Specifically, we introduce a regularization by replacing all photon wave vectors,

$$k \to h_\eta(k) \equiv k\, f_\eta(k) \,, \tag{4.175a}$$

where the regulator $h_\eta(k)$ fulfills the following conditions, $h_\eta(k) \approx k$ for $k \sim R^{-1}$, and $h_\eta(k) \to 0$ for $k \gg R^{-1}$, i.e. the regulator suppresses the contribution of those modes for which the wave length length is much smaller than the plate distance. Also, the function $f_\eta(\omega)$ should preserve basic symmetries of the photon frequencies as a function of the wave vector $\vec{k}$, such as rotational invariance. One possibility is

$$f_\eta(k) = \exp\left(-\eta\, k\right) \tag{4.175b}$$

where $\eta \ll R$ is a free parameter; the limit $\eta \to 0$ is taken after all other operations are carried out. We then have

$$E_{\rm bare}\big|_{\rm reg.} = \frac{\hbar\, c\, L^2}{8\pi^2} \int_{\mathbb{R}^2} \mathrm{d}^2 k_\parallel \left[h_\eta\left(\left| \vec{k}_\parallel \right| \right) + 2 \sum_{n_z=1}^{\infty} h_\eta \left(\sqrt{\vec{k}_\parallel^2 + \frac{n_z^2\, \pi^2}{R^2}} \right) \right] . \tag{4.176}$$

This quantity is $\mathrm{Bare\_Observable}\big|_{\rm reg.}$ in the sense of Eq. (4.161), and the regularized integrand is immediately obvious.

However, the introduction of a regularization is only half the story. In order to obtain a physically sensible result, we also need to perform a subtraction. Namely, we know that the zero-point energy of the photon modes without the plates present, is physically unobservable. Therefore, we have to subtract the energy of these modes, duly regularized in the same manner as the energy of the modes in the modified vacuum. The difference of these two terms then gives the physically observable energy shift of the vacuum, or Casimir energy, in the sense of Eq. (4.161). The counter-term thus is

$$\begin{aligned} C &= \frac{\hbar\, c\, L^2}{4\pi^2} \int_{\mathbb{R}^2} \mathrm{d}^2 k_\parallel \int_0^\infty \mathrm{d}n_z \, \sqrt{\vec{k}_\parallel^2 + \frac{n_z^2\,\pi^2}{R^2}}\,, \\ C\big|_{\mathrm{reg.}} &= \frac{\hbar\, c\, L^2}{4\pi^2} \int_{\mathbb{R}^2} \mathrm{d}^2 k_\parallel \int_0^\infty \mathrm{d}n_z \; h_\eta\left(\sqrt{\vec{k}_\parallel^2 + \frac{n_z^2\,\pi^2}{R^2}}\right) . \end{aligned} \tag{4.177}$$

We therefore have the physically observable, renormalized zero-point energy shift $E_{\mathrm{ren}}(R)$ as a function of the distance R of the plates as $E_{\mathrm{ren}}(R) = E_{\mathrm{bare}}\big|_{\mathrm{reg.}} - C\big|_{\mathrm{reg.}}$ with

$$\begin{aligned} E_{\mathrm{ren}}(R) &= \frac{\hbar\, c\, L^2}{4\pi^2} \int_{\mathbb{R}^2} \mathrm{d}^2 k_\parallel \left[\frac{1}{2}\, h_\eta\left(\left|\vec{k}_\parallel\right|\right) + \sum_{n_z=1}^\infty h_\eta\left(\sqrt{\vec{k}_\parallel^2 + \frac{n_z^2\,\pi^2}{R^2}}\right) - \int_0^\infty \mathrm{d}n_z \; h_\eta\left(\sqrt{\vec{k}_\parallel^2 + \frac{n_z^2\,\pi^2}{R^2}}\right)\right] \\ &= \frac{\hbar\, c\, L^2}{2\pi} \int_0^\infty \mathrm{d}k_\parallel \, k_\parallel \left[\frac{1}{2}\, h_\eta(k_\parallel) + \sum_{n_z=1}^\infty h_\eta\left(\sqrt{k_\parallel^2 + \frac{n_z^2\,\pi^2}{R^2}}\right) - \int_0^\infty \mathrm{d}n_z \; h_\eta\left(\sqrt{k_\parallel^2 + \frac{n_z^2\,\pi^2}{R^2}}\right)\right] . \end{aligned} \tag{4.178}$$

We now substitute

$$k_\parallel = \frac{\pi}{R}\sqrt{u}\,, \qquad u = \frac{R^2\, k_\parallel^2}{\pi^2}\,. \tag{4.179}$$

This implies the relations

$$\frac{\mathrm{d}k_\parallel}{\mathrm{d}u} = \frac{\pi}{2\,R\,\sqrt{u}}\,, \qquad k_\parallel\,\mathrm{d}k_\parallel = \frac{\pi^2}{2R^2}\,\mathrm{d}u\,, \tag{4.180a}$$

$$h_\eta(k_\parallel) = \frac{\pi}{R}\sqrt{u}\; f_\eta\left(\frac{\pi}{R}\sqrt{u}\right) , \tag{4.180b}$$

$$h_\eta\left(\sqrt{k_\parallel^2 + \frac{n^2\pi^2}{R^2}}\right) = \frac{\pi}{R}\sqrt{u+n_z^2}\; f_\eta\left(\frac{\pi}{R}\sqrt{u+n_z^2}\right) . \tag{4.180c}$$

The substitution $k_\parallel \to u$, applied to Eq. (4.178), leads to

$$\begin{aligned} \frac{E_{\mathrm{ren}}(R)}{L^2} &= \lim_{\eta\to 0} \frac{\hbar\, c\, \pi^2}{4\, R^3} \int_0^\infty \mathrm{d}u \left[\frac{1}{2}\sqrt{u}\; f_\eta\left(\frac{\pi}{R}\sqrt{u}\right) + \sum_{n_z=1}^\infty \sqrt{u+n_z^2}\; f_\eta\left(\frac{\pi}{R}\sqrt{u+n_z^2}\right)\right. \\ &\qquad \left. - \int_0^\infty \mathrm{d}n_z \; \sqrt{u+n_z^2}\; f_\eta\left(\frac{\pi}{R}\sqrt{u+n_z^2}\right)\right] \\ &= \lim_{\eta\to 0} \frac{\hbar\, c\, \pi^2}{4\, R^3} \left[\frac{1}{2}\, F_\eta(0) + \sum_{n_z=1}^\infty F_\eta(n_z) - \int_0^\infty \mathrm{d}n_z \; F_\eta(n_z)\right] . \end{aligned} \tag{4.181}$$

We absorb the integration over u into the definition of the function $F_\eta(n_z)$ by

$$F_\eta(n_z) = \int_0^\infty \mathrm{d}u\, \sqrt{u+n_z^2}\; f_\eta\left(\frac{\pi}{R}\sqrt{u+n_z^2}\right) = \int_{n_z^2}^\infty \mathrm{d}v\, \sqrt{v}\; f_\eta\left(\frac{\pi}{R}\sqrt{v}\right) . \tag{4.182}$$

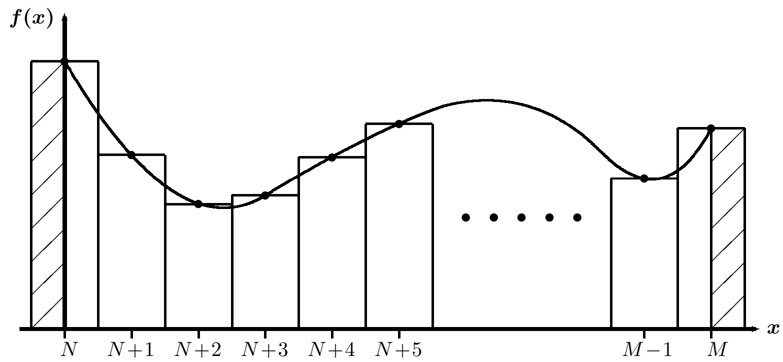


Figure 4.15: Illustration of Eq. (4.183).

Note that we have exchanged the order of the u- and n-integrations for the last term. This is possible because the regularization has led to integrals which are absolutely convergent; and this aspect is very important as it illustrates the necessity of keeping all regulators finite up until the very last steps of the calculation.

The expression in square brackets in the last line of Eq. (4.181) has the structure of a discrete sum minus a corresponding integral. The prefactor $\frac{1}{2}$ in front of the first term finds a natural, geometrical interpretation. Quite fortunately, the Euler–Maclaurin formula comes to the rescue in our quest of evaluating the tiny difference between the sum and the integral in the last line of Eq. (4.181). The Euler–Maclaurin summation formula is given in many textbooks see, e.g., Eqs. (2.01) and (2.02) on p. 285 of [F. W. J. Olver, *"Asymptotics and Special Functions"* (Academic Press, New York, 1974)]. In its full form, the Euler–Maclaurin formula (see also Fig. 4.15 reads

$$\tfrac{1}{2}f(N) + \tfrac{1}{2}f(M) + \sum_{n=N+1}^{M-1} f(n) - \int_N^M \mathrm{d}n\, f(n) = \sum_{j=1}^{q} \frac{B_{2j}}{(2j)!} \left[f^{(2j-1)}(M) - f^{(2j-1)}(N)\right] + R_q\,, \tag{4.183a}$$

where the remainder term is

$$R_q = -\frac{1}{(2q)!} \int_N^M \mathrm{d}x\, B_{2q}\big(x - [\![x]\!]\big)\, f^{(2q)}(x)\,. \tag{4.183b}$$

Here, $[\![x]\!]$ is the integral part of x, i.e., the largest integer m satisfying $m \leq x$, and $B_k(x)$ is a Bernoulli polynomial defined by the generating function [see Eq. (1.06) on p. 281 of *loc. cit.*]

$$\frac{t\exp(xt)}{\exp(t) - 1} = \sum_{m=0}^{\infty} B_m(x)\, \frac{t^m}{m!}\,, \qquad |t| < 2\pi\,. \tag{4.184}$$

The $B_m = B_m(0)$ are the Bernoulli numbers, for which we can indicate some example cases,

$$B_1 = -\frac{1}{2}\,, \quad B_2 = \frac{1}{6}\,, \quad B_3 = 0\,, \quad B_4 = -\frac{1}{30}\,, \quad B_5 = 0\,, \quad B_6 = \frac{1}{42}\,, \ldots. \tag{4.185}$$

Geometrically, it is clear that the expression $\frac{1}{2}f(N) + \frac{1}{2}f(M) + \sum_{n=N+1}^{M-1} f(n)$ approximates the integral $\int_N^M \mathrm{d}x\, f(x)$ according to the trapezoid rule, the approximation becoming exact for a linear dependence of the function f on its argument. The remainder term obtained by forming the difference is then expressed, by the Euler–Maclaurin formula, as a sum over expressions containing higher-order derivatives of the function. In practical applications, the assumption is that the remainder term R_q on the right-hand side of Eq. (4.183b) tends to zero as $q \to \infty$, and that the first few terms on the right-hand side of Eq. (4.183a) yield increasingly better approximations to the remainder term given by the difference of the sum and the integral over n.

For our case [see Eq. (4.181)], we identify $N \to 0$, $M \to \infty$, $f \to F_\eta$ and $n \to n_z$. The convergence of the integral $\int_0^\infty \mathrm{d}n_z\, F_\eta(n_z)$ is ensured by the exponential damping at large n_z given by the regulator, and the same exponential suppression of the contribution from large n_z is responsible for the fact that all derivatives of the function $F_\eta(n_z)$ vanish in the limit $n_z \to \infty$. We can thus rewrite the Eq. (4.181) considering only the derivatives of the function $F_\eta(n_z)$ at $n_z = 0$,

$$\begin{aligned} \tfrac{1}{2}F_\eta(0) + \sum_{n_z=1}^{\infty} F_\eta(n_z) - \int_0^\infty \mathrm{d}n_z\, F_\eta(n_z) &= -\sum_{j=1}^{\infty} \frac{1}{(2j)!}\, B_{2j}\, F_\eta^{(2j-1)}(0) \\ &= -\frac{1}{2!}\, B_2\, F'_\eta(0) - \frac{1}{4!}\, B_4\, F_\eta^{'''}(0) - \frac{1}{6!}\, B_6\, F_\eta^{'''''}(0) - \dots . \end{aligned} \tag{4.186}$$

We new have expressed the renormalized vacuum energy shift as

$$\frac{E_{\mathrm{ren}}(R)}{L^2} = \lim_{\eta\to 0} \frac{\hbar\, c\, \pi^2}{4\, R^3} \left[-\frac{1}{2!}\, B_2\, F'_\eta(0) - \frac{1}{4!}\, B_4\, F_\eta^{'''}(0) - \frac{1}{6!}\, B_6\, F_\eta^{'''''}(0) - \dots \right] . \tag{4.187}$$

The evaluation of the derivatives now proceeds using the representation for $F_\eta(n_z)$ given in Eq. (4.182), and we recall here for convenience this definition as well as the definition of $f_\eta(k)$ originally given in Eq. (4.175b),

$$F_\eta(n_z) = \int_{n_z^2}^\infty \mathrm{d}v\, \sqrt{v}\; f_\eta\left(\frac{\pi}{R}\sqrt{v}\right) , \qquad f_\eta\,(k) = \exp(-\eta\, k). \tag{4.188}$$

The differentiation with respect to n_z^2 can easily be rewritten into a differentiation with respect to n_z,

$$\frac{\partial F_\eta(n_z)}{\partial n_z^2} = -n_z\; f_\eta\left(\frac{\pi}{R}\, n_z\right) = \frac{1}{2n_z} \frac{\partial F_\eta(n_z)}{\partial n_z} \tag{4.189}$$

and so

$$\frac{\partial F_\eta(n_z)}{\partial n_z} = -2n_z^2\; f_\eta\left(\frac{\pi}{R}\, n_z\right) = -2n_z^2\; \exp\left(-\frac{\pi\eta}{R}\, n_z\right) . \tag{4.190}$$

This means that

$$F'_\eta(n_z) = -\,2n_z^2\; \exp\left(-\eta\, \frac{\pi}{R}\, n_z\right) , \tag{4.191a}$$

$$F_\eta^{''}(n_z) = \left(-4n_z + \frac{2n_z^2\pi\eta}{R}\right) \exp\left(-\frac{\pi\eta}{R}\, n_z\right) , \tag{4.191b}$$

$$F_\eta^{'''}(n_z) = \left(-4 + \frac{8n_z\pi\eta}{R} - \frac{2n_z^2\pi^2\eta^2}{R^2}\right) \exp\left(-\frac{\pi\eta}{R}\, n_z\right) , \tag{4.191c}$$

$$F_\eta^{''''}(n_z) = \left(\frac{12\pi\eta}{R} - \frac{12n_z\pi^2\eta^2}{R^2} + \frac{2n_z^2\pi^3\eta^3}{R^3}\right) \exp\left(-\frac{\pi\eta}{R}\, n_z\right) , \tag{4.191d}$$

$$F_\eta^{'''''}(n_z) = \left(-\frac{24\pi^2\eta^2}{R^2} + \frac{16n_z\pi^3\eta^3}{R^3} - \frac{2n_z^2\pi^4\eta^4}{R^4}\right) \exp\left(-\frac{\pi\eta}{R}\, n_z\right) . \tag{4.191e}$$

We have now performed all integrations and differentiations with a finite cutoff η. Note that both the actual zero-point energy as well as the counter term have been regularized, and the regularization parameter η can now be removed at the very end of the calculation. Namely, we can now easily read off the results

$$\lim_{\eta\to 0} F'_\eta(0) = \lim_{\eta\to 0} F''_\eta(0) = 0\,, \tag{4.192a}$$

$$\lim_{\eta\to 0} F'''_\eta(0) = -4\,, \tag{4.192b}$$

$$\lim_{\eta\to 0} F^{(j\geq 4)}_\eta(0) = 0\,. \tag{4.192c}$$

All higher derivatives vanish because further powers of η are generated by the differentiation of the exponential function.

After removing the regularization, we obtain the Casimir energy shift as

$$\frac{E_{\rm ren}(R)}{L^2} = -\frac{\hbar\, c\, \pi^2}{4\, R^3}\,\frac{1}{4!}\,B_4\,\lim_{\eta\to 0} F'''_\eta(0) = -\frac{\hbar\, c\, \pi^2}{4\, R^3}\,\frac{1}{24}\left(-\frac{1}{30}\right)(-4) = -\frac{\hbar\, c\, \pi^2}{720\, R^3}\,. \tag{4.193}$$

The energy shift is negative, corresponding to an attractive interaction. Indeed, there is an analogy to the Casimir force in sailing on the open sea. Two ships, not far apart, are known to be in danger of colliding when they approach each other too closely. The point is that in between the ships, the waves ("vacuum fluctuations") are suppressed, which is why the waves that exert forces on the left ship from port, and on the right ship from starboard, will tend to push the ships toward each other. In the same way, the Casimir energy is generated by the suppression of vacuum fluctuations in between the two plates.

The Casimir force per area (Casimir pressure) on the plate at $z = R$ is consequently negative (attractive) and amounts to

$$\frac{F_{\rm ren}(R)}{L^2} = -\frac{\partial}{\partial R}\,\frac{E_{\rm ren}(R)}{L^2} = -\frac{\hbar\, c\, \pi^2}{240\, R^4}\,. \tag{4.194}$$

This concludes our brief investigation of the Casimir effect, which, as a first hors d'œuvre toward greater endeavours, has familiarized us with the ideas of regularization and renormalization.

Chapter 5

Electromagnetic Waves in Media

5.1 Orientation

This chapter is devoted to a discussion of the optical properties of dense materials, and to their response to electromagnetic irradiation. Topics include the Sellmeier equation, the Clausius–Mosotti relation, and the Kramers–Kronig relation.

5.2 From Oscillator Strengths to Dense Materials

5.2.1 Sellmeier Equation

The propagation of waves in a medium depends on the magnetic permeability and electric permittivity functions $\mu(\omega)$ and $\epsilon(\omega)$ for the medium. We will generally deal with systems in which $\mu(\omega)$ can be approximated as having the value unity. A general form often used to approximate the electric permittivity is the Sellmeier equation, already discussed in Sec. 2.4.3,

$$\tilde{\epsilon}(\omega) = \epsilon_r(\omega)\, \epsilon_0\,, \qquad \epsilon_r(\omega) = 1 + \sum_m \frac{\mathcal{A}_m}{\omega_m^2 - \omega^2 - \mathrm{i}\gamma_m \omega}\,. \tag{5.1}$$

Here, $\mathcal{A}_m = \Omega_m^2$ has the dimension of the square of a frequency. The relative permittivity ϵ_r of a sample is dimensionless. The real and imaginary parts of this equation are as follows,

$$\mathrm{Re}\, \epsilon_r(\omega) = 1 + \sum_m \frac{(\omega_m^2 - \omega^2)\, \mathcal{A}_m}{(\omega^2 - \omega_m^2)^2 + \omega^2\, \gamma_m^2} \sim 1 - \sum_m \frac{\mathcal{A}_m}{\omega^2}\,, \quad \omega \to \infty\,, \tag{5.2}$$

and

$$\mathrm{Im}\, \epsilon_r(\omega) = \sum_m \frac{\omega\, \gamma_m\, \mathcal{A}_m}{(\omega^2 - \omega_m^2)^2 + \omega^2\, \gamma_m^2} \sim \sum_m \frac{\gamma_m\, \mathcal{A}_m}{\omega^3}\,, \quad \omega \to \infty\,. \tag{5.3}$$

The real part is an even function of ω, the imaginary part is an odd function of ω. The structure of the dielectric function given by Eq. (5.1) is quite universal.

This equation is based on the assumption that the system behaves as a set of resonances. In addition to the atomic physics model discussed in Sec. 2.4.3, we here consider electromagnetic waves in ionic crystals.

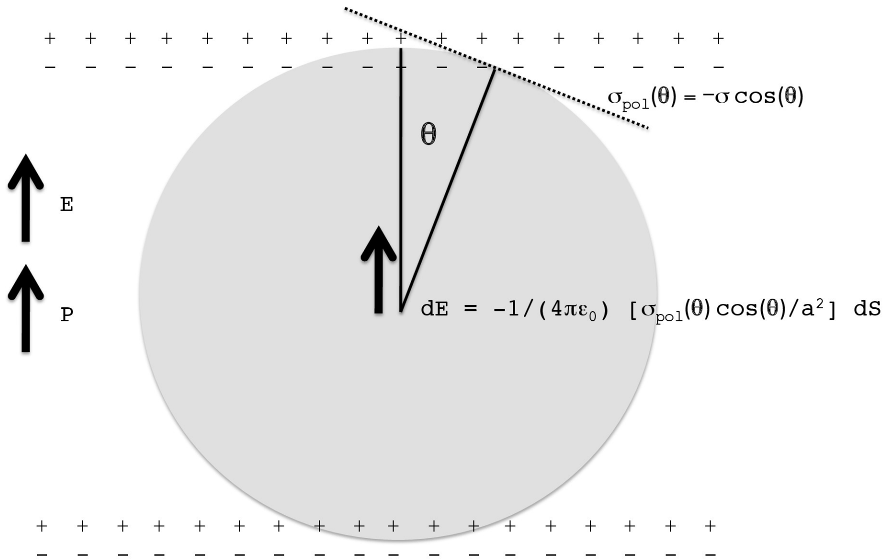


Figure 5.1: Illustration of Eq. (5.12) and (5.13).

The dielectric function is determined by the ionic displacements and the polarizabilities of the ions. The second kind of systems are electrical conductors, either metals or plasmas. In these systems, the frequency-dependent conductivity determines the dielectric function.

The analysis of the models will be followed by a discussion of wave packets in dispersive materials. This discussion uses the permittivities obtained for the two models. The propagation of electromagnetic waves in a medium involves the index of refraction of the medium. From the deduced analytic properties of the complex index of refraction we will obtain the Kramer-Kronig relationship between the real and imaginary parts of the index of refraction.

5.2.2 Dielectric Constant for Gases and Dense Materials

For a dilute gas, the dielectric constant fulfills

$$\tilde{\epsilon}(\omega) = \tilde{\epsilon}_r(\omega)\,\epsilon_0 \approx \epsilon_0\,, \tag{5.4}$$

because the deviation from the vacuum dielectric constant is small. We recall Eq. (2.215),

$$\tilde{\epsilon}_r(\omega) = 1 + \frac{N_V}{\epsilon_0}\,\alpha(\omega) = 1 + \frac{N_V}{\epsilon_0}\,\sum_n \frac{f_{n0}}{E_{n0}^2 - \mathrm{i}\,\Gamma_n\,\hbar\,\omega - (\hbar\omega)^2}\,, \tag{5.5}$$

where the dipole oscillator strengths have a dimension of $e^2\,a_0^2\,E_{n0}$ with an obvious meaning of the symbols. We have thus expressed $\tilde{\epsilon}(\omega)$ in terms of the polarizabilities of the individual gas atoms. We also recall that the polarization is given as

$$\vec{P}(\vec{r}) = \sum_n \vec{p}_n\,f(\vec{r} - \vec{r}_n)\,, \tag{5.6}$$

where f constitutes the test function for the averaging volume V over which the macroscopic Maxwell equations are determined. The distribution function $f(\vec{r} - \vec{r}_n)$ is normalized as

$$\int_V \mathrm{d}^3r\,f(\vec{r} - \vec{r}_n) = 1\,. \tag{5.7}$$

Thus, f has dimension of inverse volume. It is advantageous to choose the reference volume V, as a rectangular volume of dimensions Δx, Δy, and Δz. We choose Δz to be the distance of the two charges that represent the dipole. If the dipoles are distributed uniformly in the test volume, then

$$\vec{P}(\vec{r}) = \langle \vec{p} \rangle \, \frac{N}{V} = N_V \, \langle \vec{p} \rangle \, . \tag{5.8}$$

Let us suppose that the dipoles are oriented in the z direction. Then, a cut through the sample, in the xy plane, with N atoms in the volume, will result in a surface charge density

$$\sigma = N \, \frac{q}{\Delta x \, \Delta y} = \frac{N}{\Delta x \, \Delta y \, \Delta z} \, (q \Delta z) = N_V \, |\langle \vec{p} \rangle| = |\vec{P}(\vec{r})| \, . \tag{5.9}$$

If the surface is tilted, then the appropriate modification is

$$\sigma = |\vec{P}(\vec{r})| \, \cos\theta \, . \tag{5.10}$$

The sign and orientation of $\vec{P}(\vec{r})$ then have to be determined based on an additional consideration.

We recall that for a uniform medium, the polarization

$$\vec{P}(\vec{r}, \omega) = \epsilon_0 \, (\epsilon_r(\omega) - 1) \, \vec{E}(\vec{r}, \omega) \tag{5.11}$$

is aligned and parallel to the applied electric field. We have $\epsilon_0 \, \vec{\nabla} \cdot \vec{E}(\vec{r}, \omega) = \rho_{\rm total}(\vec{r}, \omega)$ and $\vec{\nabla} \cdot \vec{P}(\vec{r}, \omega) = -\rho_{\rm bound}(\vec{r}, \omega)$ so that $\vec{\nabla} \cdot \vec{D}(\vec{r}, \omega) = \epsilon_0 \, \vec{\nabla} \cdot \vec{E}(\vec{r}, \omega) + \vec{\nabla} \cdot \vec{P}(\vec{r}, \omega) = \rho_{\rm total}(\vec{r}, \omega) - \rho_{\rm bound}(\vec{r}, \omega) = \rho_{\rm free}(\vec{r}, \omega)$.

For a dense material such as a solid, the dielectric constant $\epsilon_r(\omega)$ may substantially deviate from unity. In this case, it becomes necessary to invoke a more sophisticated treatment. Let us draw a sphere of radius a about the point $\vec{r}$ in the dense sample. Let us assume, furthermore, that the polarization points into the $+z$ direction, and the applied electric field also points along $+z$. We cut through the dipole layers in the middle of the dipoles, separating the dipoles, and ignore the outer portion of the sphere as well as the rest of the bulk medium in the following. Then, the surface charge density at an angle θ with respect to the symmetry axis of the polarization (the $+z$ axis is parallel, not anti-parallel, to $\vec{P}$) is

$$\sigma_{\rm pol}(\theta) = -P \, \cos\theta \, , \tag{5.12}$$

where θ is the polar angle with respect to the $+z$ axis (see also Fig. 5.1). The sign in $P \, \cos\theta$ is evident from a geometrical consideration. Let us assume that the dipoles are oriented so that the positive charges are to the top and the negative ones to the bottom. The polarization vector points upward. We use the center of the sphere as an anchor point. This center is below the upper surface of the sphere. Cutting through the sample, positive charge is left on the upper side of the sphere, i.e., on the outer rim of the cut-out sphere. At the same time, $\theta = 0$. What counts for us is the negative charge on the lower side of the cut. This justifies the negative sign in Eq. (5.12).

Now, imagine that we cut out a sphere of radius a about our reference point $\vec{r}$. The field due to the infinitesimal surface charge element of radius a, generated at the reference point $\vec{r}$, amounts to

$$\mathrm{d}E_z = -\frac{1}{4\pi\epsilon_0} \, \frac{\sigma_{\rm pol}(\theta) \, \cos\theta}{a^2} \, \mathrm{d}S = \frac{1}{4\pi\epsilon_0} \, \frac{+P \, \cos^2\theta}{a^2} \, \mathrm{d}S \, , \tag{5.13}$$

because the positive charges at the top of the sphere lead to an electric field that points downward. The field generated by the polarized constituents at the point $\vec{r}$ is

$$E_{\rm pol} = E_z \, \hat{\mathrm{e}}_z = \frac{1}{4\pi\epsilon_0} \int \frac{P \, \cos^2\theta}{a^2} \, \mathrm{d}S = \frac{P}{4\pi\epsilon_0} \, \frac{1}{a^2} \, \frac{4\pi a^2}{3} = \frac{P}{3\epsilon_0} \, , \qquad \vec{E}_{\rm pol}(\vec{r}) = \frac{\vec{P}(\vec{r})}{3\epsilon_0} \, . \tag{5.14}$$

The total field of the remaining dipoles within the sphere of radius R is

$$\sum_i \vec{E}_i = \frac{1}{4\pi\epsilon_0} \sum_i \left(\frac{3(\vec{p}_i \cdot \vec{r}_i)\, \vec{r}_i}{r_i^5} - \frac{\vec{p}_i}{r_i^3} \right) \tag{5.15}$$

where i enumerates the molecules, and $\vec{r}_i$ is the direction vector from the location of the ith molecule to $\vec{r}$. Provided the entire sample is z polarized, we have

$$3(\vec{p}_i \cdot \vec{r}_i)\, \vec{r}_i \to 3\hat{\mathrm{e}}_z\, p_i\, z_i^2 \to \frac{1}{3} \times 3\hat{\mathrm{e}}_z\, p_i\, r_i^2\,, \tag{5.16}$$

where in the last step, we have performed the angular average $z_i^2 \to r_i^2/3$. In summary,

$$\sum_i \vec{E}_i = \vec{0}\,. \tag{5.17}$$

The polarization charge is exclusively given by the integral (5.14). Letting the radius of the sphere become very large, we see that the result (5.14) holds universally.

In summary, we have derived the following result: Suppose an atom sits in a dense, homogeneously polarized sample with polarization $\vec{P}$. We cannot use the dielectric displacement $\vec{D}$ in order to evaluate the local field felt by the atom at the reference point $\vec{r}$. Rather, there is a geometric factor which enhances the local field by the effect of the contributing polarization, i.e., by a term $\vec{P}(\vec{r})/(3\epsilon_0)$.

The local field at the point $\vec{r}$ is thus found by integrating over the polarizability of the atoms near the surface. Let $\vec{E}_{\mathrm{ext}}(\vec{r})$ be the "external" field applied to the sample, as depicted in Fig. 5.1. One has the total electric field, which is also equal to the local electric field, as follows,

$$\vec{E}_{\mathrm{loc}}(\vec{r}) = \vec{E}_{\mathrm{avg}}(\vec{r}) + \vec{E}_{\mathrm{pol}}(\vec{r}) = \vec{E}_{\mathrm{avg}}(\vec{r}) + \frac{1}{3\,\epsilon_0}\, \vec{P}(\vec{r})\,. \tag{5.18}$$

On the other hand, we have, by the definition of the polarization,

$$\vec{P}(\vec{r}) = \epsilon_0\, [\epsilon_r(\omega) - 1]\, \vec{E}_{\mathrm{avg}}(\vec{r})\,. \tag{5.19}$$

So,

$$\vec{E}_{\mathrm{loc}}(\vec{r}) = \vec{E}_{\mathrm{avg}}(\vec{r}) + \frac{1}{3\,\epsilon_0}\, \epsilon_0\, [\epsilon_r(\omega) - 1]\, \vec{E}_{\mathrm{avg}}(\vec{r}) = \frac{\epsilon_r(\omega) + 2}{3}\, \vec{E}_{\mathrm{avg}}(\vec{r})\,. \tag{5.20}$$

This leads to the somewhat paradoxical result that the local field is larger than that the average field, in the dense polarized sample, even if the polarization effects are otherwise known to counteract the "driving, external" field [see also the discussion surrounding Eq. (1.45)*].* We can now use the definition of the polarization and relate it to the electric field,

$$\epsilon_0\, [\epsilon_r(\omega) - 1]\, \vec{E}_{\mathrm{avg}}(\vec{r}) = \vec{P}(\vec{r}) = N_V\, \alpha(\omega)\, \vec{E}_{\mathrm{loc}}(\vec{r}) = N_V\, \alpha(\omega)\, \frac{\epsilon_r(\omega) + 2}{3}\, \vec{E}_{\mathrm{avg}}(\vec{r})\,. \tag{5.21}$$

Comparing the first to the last expression in the above equation, we find that

$$\frac{\epsilon_r(\omega) - 1}{\epsilon_r(\omega) + 2} = \frac{N_V\, \alpha(\omega)}{3\,\epsilon_0}\,, \tag{5.22}$$

which is the Clausius-Mosotti equation. It is easy to show that

$$\frac{\mathrm{d}\epsilon_r}{\mathrm{d}N_V} = \frac{(\epsilon_r(\omega) - 1)\, (\epsilon_r(\omega) + 2)}{3\, N_V}\,. \tag{5.23}$$

A remark is in order: In terms of the averaging of the fields, with the equation

$$\vec{D}(\vec{r},t) = \epsilon_0 \, \langle \vec{e}(\vec{r},t) \rangle + \sum_n \vec{P}_n \, f(\vec{r}-\vec{r}_n) \,, \tag{5.24}$$

one might ask why $\vec{E}_{\rm avg}(\vec{r})$ in Eq. (5.18) could not be interpreted as the average $\langle \vec{e}(\vec{r}) \rangle$, where $\vec{e}(r) = \vec{E}_{\rm loc}(\vec{r})$ is the local field. Under this interpretation, one would necessarily have the relationship $\vec{E}(\vec{r}) = \langle \vec{e}(r) \rangle = \langle \vec{E}_{\rm loc}(\vec{r}) \rangle$, which, together with $\vec{E}_{\rm avg}(\vec{r}) = \langle \vec{E}(r) \rangle$, would lead to the paradoxical result $\vec{P}(\vec{r}) = \vec{0}$ when Eq. (5.18) is averaged over a sample volume.

5.3 Dielectric Constant for Ionic Crystals

5.3.1 Basic Formulas

We analyze the frequency-dependent dielectric constant of a typical solid, assumed to be a crystal. The constituent atoms are assumed to be ions, with polarizabilties $\alpha^+(\omega)$ and $\alpha^-(\omega)$. In addition, we have to take into account the crystal vibrations which can be excited when the ions in the solid are set in motion by the interaction with an external electric field.

Let us assume that the ionic vibrations can be described by a motion of the ions in a harmonic potential, with resonant frequency $\bar{\omega}$ for the crystal vibrations. When a charged particle of mass m, bound in a harmonic potential such as an ion in a crystal, with resonant frequency $\bar{\omega}$, is subject to a perturbation, the polarizability is given by the following expression,

$$\alpha(\omega) = \frac{q^2}{m} \frac{1}{\bar{\omega}^2 - \omega^2} \,. \tag{5.25}$$

The total polarizability for an ion in an ionic crystal, averaged over singly charged positive and negative ions ($|q| = e$), is given by

$$\alpha(\omega) = \frac{1}{2} \, \left(\alpha^+(\omega) + \alpha^-(\omega)\right) + \frac{e^2}{m(\bar{\omega}^2 - \omega^2)} \,. \tag{5.26}$$

We here assume that the sample is globally neutral, and define N_V to be the molar concentration of positive+negative ions, i.e., $N_V = N/V$ where N increases by one if we count either a negative or a positive ion. The Clausius-Mosotti equation for an ionic crystal thus is

$$\frac{\epsilon_r(\omega) - 1}{\epsilon_r(\omega) + 2} = \frac{N_V}{3\epsilon_0} \left(\frac{1}{2} \, \alpha^+(\omega) + \frac{1}{2} \, \alpha^-(\omega) + \frac{e^2}{m \, (\bar{\omega}^2 - \omega^2)} \right) \,. \tag{5.27}$$

Here, m is the "mass" associated with the collective phonon frequency of the crystal.

The individual polarizabilities $\alpha^+(\omega)$ and $\alpha^-(\omega)$ deviate from the static polarizabilities $\alpha^+(0)$ and $\alpha^-(0)$ only when the angular frequency ω is in the range of a typical atomic/ionic transition frequency $\omega_{\rm at}$ of the ion, i.e., in the optical range. The resonant frequency $\bar{\omega}$ of the crystal lattice typically is much lower. For illustration, we consider ionic crystals such as the alkali-halides whose crystal lattice is cubic. Note that electromagnetic fields only interact with the "optical modes" of the crystal. These are the modes in which the positive and negative ions move in opposite directions. The dielectric function of cubic crystals is independent of the direction of the electric field and can be characterized by three parameters, the static dielectric constant ϵ_s, the optical dielectric constant $\epsilon_{\rm opt}$, and a characteristic resonant frequency ω_T. For reference, typical resonant frequencies are in the range $0.02\,{\rm eV}$—$0.04\,{\rm eV}$, optical frequencies are in the range $2\,{\rm eV}$ to $3\,{\rm eV}$, and the electronic interband transitions begin around 4 or $5\,{\rm eV}$.

The static relative dielectric constant of the solid therefore fulfills, with $\epsilon_r^{(0)} = \epsilon_r(\omega = 0)$,

$$\frac{\epsilon_r^{(0)} - 1}{\epsilon_r^{(0)} + 2} = \frac{N_V}{3\epsilon_0} \left(\frac{1}{2}\,\alpha^+(0) + \frac{1}{2}\,\alpha^-(0) + \frac{e^2}{m\,\bar{\omega}^2} \right) . \tag{5.28}$$

For intermediate angular frequencies $\bar{\omega} \ll \omega = \omega_{\rm int} \ll \omega_{\rm at}$, one has, with $\epsilon_r^{(\infty)} \equiv \epsilon_r(\omega_{\rm int})$ in the intermediate frequency range,

$$\frac{\epsilon_r^{(\infty)} - 1}{\epsilon_r^{(\infty)} + 2} = \frac{N_V}{3\epsilon_0} \left(\frac{1}{2}\alpha^+(0) + \frac{1}{2}\alpha^-(0) \right) . \tag{5.29}$$

So,

$$\frac{\epsilon_r^{(0)} - 1}{\epsilon_r^{(0)} + 2} - \frac{\epsilon_r^{(\infty)} - 1}{\epsilon_r^{(\infty)} + 2} = \frac{N_V}{3\epsilon_0}\,\frac{e^2}{m\,\bar{\omega}^2} \tag{5.30}$$

Finally, we can find a functional relationship between the dielectric constants in different frequency ranges,

$$\begin{aligned}\frac{\epsilon_r(\omega) - 1}{\epsilon_r(\omega) + 2} &\approx \frac{N_V}{3\epsilon_0} \left(\frac{1}{2}\alpha^+(0) + \frac{1}{2}\alpha^-(0) \right) + \frac{N_V}{3\epsilon_0}\,\frac{e^2}{m\,\bar{\omega}^2}\,\frac{1}{1 - \omega^2/\bar{\omega}^2} \\ &= \frac{\epsilon_r^{(\infty)} - 1}{\epsilon_r^{(\infty)} + 2} + \left(\frac{\epsilon_r^{(0)} - 1}{\epsilon_r^{(0)} + 2} - \frac{\epsilon_r^{(\infty)} - 1}{\epsilon_r^{(\infty)} + 2} \right) \frac{1}{1 - \omega^2/\bar{\omega}^2}\end{aligned} \tag{5.31}$$

A brief calculation, solving for $\epsilon_r(\omega)$, then shows that

$$\epsilon_r(\omega) = \epsilon_r^{(\infty)} + \frac{\epsilon_r^{(0)} - \epsilon_r^{(\infty)}}{1 - \omega^2/\omega_{\rm T}^2} , \qquad \omega_{\rm T}^2 = \bar{\omega}^2\,\frac{\epsilon_r^{(\infty)} + 2}{\epsilon_r^{(0)} + 2} . \tag{5.32}$$

As of now, $\omega_{\rm T}$ is nothing but a definition. For the derivations reported in the following, this relation can be rewritten as

$$\epsilon_r(\omega) = \epsilon_r^{(\infty)} + \omega_{\rm T}^2\,\frac{\epsilon_r^{(0)} - \epsilon_r^{(\infty)}}{\omega_{\rm T}^2 - \omega^2} \quad \to \quad \epsilon_r(\omega) = \epsilon_r^{(\infty)} + \omega_{\rm T}^2\,\frac{\epsilon_r^{(0)} - \epsilon_r^{(\infty)}}{\omega_{\rm T}^2 - \omega^2 - {\rm i}\,\gamma\,\omega} . \tag{5.33}$$

In the latter form, we introduce an infinitesimal imaginary part in order to allow for damping. We have

$$\lim_{\omega\to\infty} \epsilon_r(\omega) = \epsilon_r^{(\infty)} , \qquad \lim_{\omega\to 0} \epsilon_r(\omega) = \epsilon_r^{(0)} , \qquad \epsilon_r^{(0)} > \epsilon_r^{(\infty)} . \tag{5.34}$$

We will find that $\omega_{\rm T}$ can be identified as the transverse optical phonon frequency for the crystal. A damping factor has been included to indicate what a real dielectric function might look like. This is an approximation to the Sellmeier equation, given in Eq. (5.1), in a frequency range near a particular resonant term (that of the transverse optical phonon frequency) and far from any other resonance. An illustration is given in Fig. 5.2. Our model (5.33) is compatible with the general ansatz (5.1) in the case of a single, isolated resonance frequency $\bar{\omega}$.

5.3.2 Longitudinal and Transverse Optical Crystal Vibrations

A very interesting observation can be made regarding the relation of the frequency of longitudinal versus transverse vibrations of a crystal lattice. We shall attempt to justify the existence of two branches of the optical spectrum of a crystal (two possible values of ω for given k), by investigating perfect longitudinal and transverse modes. In our study of wave guides, we have already seen that such considerations should

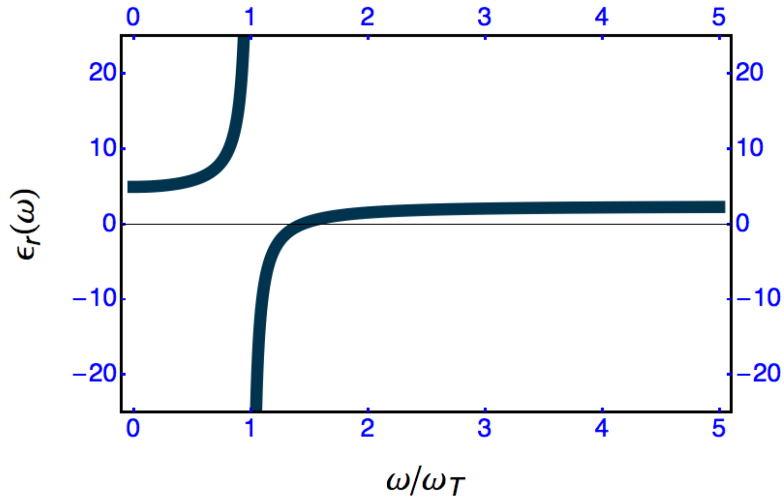


Figure 5.2: A typical example for a ionic crystal is KBr, with $\epsilon_r^{(0)} = 4.90$, and $\epsilon_r^{(\infty)} = 2.34 < \epsilon_r^{(0)}$, and $\omega_{\rm T} = 2.26 \times 10^{13}\,{\rm rad/s}$, at energy $\hbar\omega_{\rm T} = 14.3\,{\rm meV}$, without damping. The abscissa is scaled to the angular frequency $\omega_{\rm T}$.

be taken with a grain of salt (waveguide modes entail both longitudinal as well as transverse modes of the field), but they are still useful.

Let us assume that for a traveling electromagnetic wave of wave vector $\vec{k}$, all $\vec{D}$, $\vec{E}$ and $\vec{P}$ have the same dependence on $\vec{r}$, namely, they are proportional to $\exp({\rm i}\vec{k}\cdot\vec{r})$. In the absence of free charges, we have

$$\vec{k}\cdot\vec{D} = \vec{0} \qquad \Rightarrow \qquad \vec{D} = \vec{0} \qquad \text{or} \qquad \vec{D},\ \vec{E},\ \vec{P} \perp \vec{k}\ \text{(transverse mode)}\,. \tag{5.35}$$

This is the condition for a normal, transverse mode. Let us also assume, hypothetically at first, that a mode exists whose electric field actually is aligned with the propagation direction, i.e., $\vec{k}\times\vec{E} = \vec{0}$. Then,

$$\vec{k}\times\vec{E} = \vec{0} \qquad \Rightarrow \qquad \vec{E} = \vec{0} \qquad \text{or} \qquad \vec{D},\ \vec{E},\ \vec{P} \parallel \vec{k}\ \text{(longitudinal mode)}\,, \tag{5.36}$$

If a longitudinal mode is realized, then $\vec{D}$, $\vec{E}$, $\vec{P} \parallel \vec{k}$. In that case, however, since still we have $\vec{k}\cdot\vec{D} = 0$, and since the mode is not transverse, we must necessarily have $\vec{D} = \vec{0}$, which implies

$$\epsilon_r(\omega_{\rm L}) = 0 \qquad \text{(longitudinal mode)}\,. \tag{5.37}$$

Another way to see this is as follows. For a harmonically oscillating wave, we can replace $\vec{k} \to -{\rm i}\,\vec{\nabla}$. On the one hand, we have $\vec{\nabla}\times\vec{E} = \vec{0}$. But this means, by Faraday's law, that $\vec{B}$ cannot change over time and can thus be assumed to be equal to zero. But then, the Ampere–Maxwell law, in the absence of free currents, can only be satisfied if $\vec{D} = \vec{0}$ and thus $\epsilon_r(\omega_{\rm L}) = 0$.

Conversely, in a transverse mode, the polarization $\vec{P}$ is perpendicular to $\vec{k}$. In view of $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$, we can formulate a condition for transverse modes at resonance, as follows. In an undamped oscillation, at exact resonance, even a small $\vec{E}$ field would lead to a huge displacement of the charges of the crystal (again, we are considering the limit of an oscillator model without damping). In that case, the crystal would burst, we would have $-\partial\vec{B}/\partial t = \vec{0}$, and therefore $\vec{k}\times\vec{E} = \vec{0}$. Now, at exact resonance for the transverse modes, since $\vec{E}$ is parallel to $\vec{P}$ and also $\vec{k}\times\vec{E} = \vec{0}$, we have the seemingly contradictory requirement on $\vec{E}$ to be simultaneously perpendicular to $\vec{k}$ (because it is parallel to $\vec{P}$), and also, parallel to $\vec{k}$ because $\vec{k}\times\vec{E} = \vec{0}$. So, $\vec{E}$ must vanish for a transverse oscillation of the crystal at exact, undamped resonance, and since the polarization does not vanish while the electric field vanishes, we must have it that

$$\epsilon_r(\omega_\mathrm{T}) = \infty \qquad \text{(transverse mode)}\,. \tag{5.38}$$

As this condition is fullfilled for $\omega = \omega_\mathrm{T}$ in our Eq. (5.33) (in its undamped version), we can therefore *a posteriori* identify ω_T with the frequency of transverse oscillations of the crystal. Solving Eq. (5.33) for $\epsilon_r(\omega_\mathrm{L}) = 0$, we find that the solution is given by

$$\omega_\mathrm{L} = \sqrt{\frac{\epsilon_r^{(0)}}{\epsilon_r^{(\infty)}}}\,\omega_\mathrm{T}\,, \qquad \omega_\mathrm{L} > \omega_\mathrm{T}\,. \tag{5.39}$$

This is known as the Lyddane-Sachs-Teller relation. For KBr, one finds $\omega_\mathrm{L} = 3.14 \times 10^{13}\,\mathrm{rad/s}$ and $(\omega_\mathrm{L}/\omega_\mathrm{T})^2 = 1.9$. The Lyddane-Sachs-Teller relation also implies that in the first approximation, the longitudinal optical modes of vibration of a crystal have a constant frequency independent of the wavelength of the charge density wave.

5.3.3 Transverse Electromagnetic Waves in the Crystal

In order to search for transverse modes, we consider the propagation of a transverse electromagnetic wave in the crystal. In the following, we assume for convenience that $\mu_r(\omega) = 1$. Since we seek transverse modes, we only consider electric fields satisfying $\vec{\nabla} \cdot \vec{E}(\vec{r}, t) = 0$. Then,

$$\begin{aligned}
&\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}(\vec{r},t)\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}(\vec{r},t)\right) - \vec{\nabla}^2 \vec{E}(\vec{r},t) \\
&\qquad \Rightarrow -\vec{\nabla}^2 \vec{E}(\vec{r},t) = \vec{\nabla} \times \left(-\frac{\partial}{\partial t}\vec{B}(\vec{r},t)\right) = -\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{B}(\vec{r},t)\right) = -\mu_0\,\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{H}(\vec{r},t)\right) \\
&\qquad \Rightarrow -\vec{\nabla}^2 \vec{E}(\vec{r},t) = -\mu_0\,\frac{\partial^2}{\partial t^2}\,\vec{D}(\vec{r},t) - \mu_0\,\frac{\partial}{\partial t}\vec{J}_0(\vec{r},t)\,.
\end{aligned} \tag{5.40}$$

If there are no external current sources ($\vec{J}_0 = 0$), then a Fourier transformation to frequency space leads to

$$\begin{aligned}
-\vec{\nabla}^2 \vec{E}(\vec{r},\omega) &= -\epsilon_r(\omega)\,[-\mathrm{i}\omega]^2\,(\epsilon_0\,\mu_0)\,\vec{E}(\vec{r},\omega) \\
&= \epsilon_r(\omega)\,\frac{\omega^2}{c^2}\,\vec{E}(\vec{r},\omega)\,,
\end{aligned} \tag{5.41}$$

or, in wave vector space,

$$k^2\,\vec{E}\left(\vec{k},\omega\right) = \epsilon_r(\omega)\,\frac{\omega^2}{c^2}\,\vec{E}\left(\vec{k},\omega\right) \tag{5.42}$$

so that

$$\left(k^2 - \epsilon_r(\omega)\,\frac{\omega^2}{c^2}\right)\,\vec{E}\left(\vec{k},\omega\right) = \vec{0}\,. \tag{5.43}$$

Thus, for a transverse electromagnetic wave to propagate in the medium, the wave vector and angular frequency must satisfy the

Dispersion Relation:
$$k^2 = \epsilon_r(\omega)\,\frac{\omega^2}{c^2}\,. \tag{5.44}$$

We can now see how the longitudinal and the optical branch of the phonon spectrum emerge: For small $\omega \to 0$, we can approximate $\epsilon_r(\omega) \to \epsilon_r^{(0)}$, and $k \approx \epsilon_r^{(0)}\,\omega/c$. Near the vicinity of $\omega \approx \omega_\mathrm{L}$, we have $k \approx 0$, because $\epsilon_r(\omega \approx \omega_{\rm L}) \approx 0$. For given k, the two possible values of ω can be found by solving a quadratic equation.

Using the dielectric function without damping, we obtain

$$\epsilon_r(\omega) = \epsilon_r^{(\infty)} + \omega_{\rm T}^2 \, \frac{\epsilon_r^{(\infty)} - \epsilon_r^{(0)}}{\omega^2 - \omega_{\rm T}^2} = \epsilon_r^{(\infty)} \frac{\omega^2 - \omega_{\rm T}^2}{\omega^2 - \omega_{\rm T}^2} + \omega_{\rm T}^2 \, \frac{\epsilon_r^{(\infty)} - \epsilon_r^{(0)}}{\omega^2 - \omega_T^2}$$

$$= \omega_{\rm T}^2 \, \frac{\epsilon_r^{(\infty)}\omega^2/\omega_{\rm T}^2 - \epsilon_r^{(\infty)}}{\omega^2 - \omega_{\rm T}^2} + \omega_{\rm T}^2 \, \frac{\epsilon_r^{(\infty)} - \epsilon_r^{(0)}}{\omega^2 - \omega_T^2}$$

$$= \omega_{\rm T}^2 \, \frac{\epsilon_r^{(\infty)}\omega^2/\omega_{\rm T}^2 - \epsilon_r^{(0)}}{\omega^2 - \omega_{\rm T}^2} = \frac{\epsilon_r^{(\infty)} \, \omega^2 - \omega_{\rm T}^2 \, \epsilon_r^{(0)}}{\omega^2 - \omega_{\rm T}^2} \, . \tag{5.45}$$

With the help of the dispersion relation, we get

$$\left(\frac{k\;c}{\omega_{\rm T}}\right)^2 = \epsilon_r(\omega) \, \frac{\omega^2}{\omega_{\rm T}^2} = \frac{\epsilon_r^{(\infty)} \left[\dfrac{\omega^2}{\omega_{\rm T}^2}\right]^2 - \epsilon_r^{(0)} \, \dfrac{\omega^2}{\omega_{\rm T}^2}}{\dfrac{\omega^2}{\omega_{\rm T}^2} - 1} \, . \tag{5.46}$$

Letting $x = \omega^2/\omega_T^2$, we have

$$\left(\frac{k\;c}{\omega_{\rm T}}\right)^2 = \frac{\epsilon_r^{(\infty)} \; x^2 - \epsilon_r^{(0)} \; x}{x - 1} \, . \tag{5.47}$$

Multiplying both sides of this equation by $x-1$, one obtains

$$\epsilon_r^{(\infty)} \; x^2 - \left[\epsilon_r^{(0)} + \left(\frac{kc}{\omega_T}\right)^2\right] \; x + \left(\frac{kc}{\omega_T}\right)^2 = 0 \, . \tag{5.48}$$

Very easily, the solution of this quadratic equation is found to be

$$x = \frac{\omega^2}{\omega_{\rm T}^2} = \frac{1}{2\epsilon_r^{(\infty)}} \left(\epsilon_r^{(0)} + \left(\frac{kc}{\omega_{\rm T}}\right)^2\right) \pm \frac{1}{2\epsilon_r^{(\infty)}} \left\{\left[\epsilon_r^{(0)} + \left(\frac{kc}{\omega_T}\right)^2\right]^2 - 4\epsilon_r^{(\infty)} \left(\frac{kc}{\omega_T}\right)^2\right\}^{1/2} \, . \tag{5.49}$$

In order to explore the limit $kc/\omega_{\rm T} \to 0$, we can write

$$x = \frac{\omega^2}{\omega_{\rm T}^2} = \frac{\epsilon_r^{(0)} + \left(\frac{kc}{\omega_{\rm T}}\right)^2}{2\epsilon_r^{(\infty)}} \left[1 \pm \left\{1 - \frac{4\epsilon_r^{(\infty)} \left(\frac{kc}{\omega_{\rm T}}\right)^2}{\left(\epsilon_r^{(0)} + \left(\frac{kc}{\omega_T}\right)^2\right)^2}\right\}^{1/2}\right]$$

$$\approx \frac{\epsilon_r^{(0)} + \left(\frac{kc}{\omega_T}\right)^2}{2\epsilon_r^{(\infty)}} \left[1 \pm \left\{1 \mp \frac{1}{2} \frac{4\epsilon_r^{(\infty)}}{2\left(\epsilon_r^{(0)}\right)^2} \left(\frac{kc}{\omega_{\rm T}}\right)^2\right\}\right] \, , \qquad \frac{kc}{\omega_{\rm T}} \to 0 \, . \tag{5.50}$$

Therefore, as $kc/\omega_T \to 0$, there are two branches,

$$\left.\frac{\omega^2}{\omega_T^2}\right|_{\text{longt.}} \approx \frac{\epsilon_r^{(0)} + \left(\frac{kc}{\omega_{\rm T}}\right)^2}{2\epsilon_r^{(\infty)}} \times 2 = \frac{\epsilon_r^{(0)}}{\epsilon_r^{(\infty)}} \left(1 + \frac{\epsilon_r^{(\infty)}}{\epsilon_r^{(0)}} \left(\frac{kc}{\omega_{\rm T}}\right)^2\right) \approx \frac{\omega_{\rm L}^2}{\omega_T^2}\,, \qquad \frac{kc}{\omega_T} \to 0\,, \tag{5.51}$$

$$\left.\frac{\omega^2}{\omega_{\rm T}^2}\right|_{\text{trans.}} \approx \frac{\epsilon_r^{(0)} + \left(\frac{kc}{\omega_{\rm T}}\right)^2}{2\epsilon_r^{(\infty)}} \, \frac{1}{2} \, \frac{4\,\epsilon_r^{(\infty)} \left(\frac{kc}{\omega_{\rm T}}\right)^2}{\left(\epsilon_r^{(0)}\right)^2}$$

$$= \left(\frac{kc}{\sqrt{\epsilon_r^{(0)}}\omega_{\rm T}}\right)^2 \left(1 + \frac{1}{\epsilon_r^{(0)}} \left(\frac{kc}{\omega_{\rm T}}\right)^2\right) \approx \left(\frac{kc}{\sqrt{\epsilon_r^{(0)}}\omega_{\rm T}}\right)^2, \qquad \frac{kc}{\omega_T} \to 0\,. \tag{5.52}$$

For transverse waves, the initial slope is

$$\omega = \frac{c}{\sqrt{\epsilon_r^{(0)}}}\, k \qquad \text{(transverse waves)}\,. \tag{5.53}$$

The quantity

$$\left(4\epsilon_r^{(\infty)} \left(\frac{kc}{\omega_{\rm T}}\right)^2\right) \left(\epsilon_r^{(0)} + \left(\frac{kc}{\omega_T}\right)^2\right)^{-2} \to 0\,, \qquad \frac{kc}{\omega_{\rm T}} \to \infty\,, \tag{5.54}$$

is very small for both large and small k. Thus, we can approximate, for large k,

$$x = \frac{\omega^2}{\omega_{\rm T}^2} \approx \frac{\epsilon_r^{(0)} + \left(\frac{kc}{\omega_T}\right)^2}{2\epsilon_r^{(\infty)}} \left[1 \pm \left\{1 \mp \frac{1}{2} \, \frac{4\epsilon_r^{(\infty)} \left(\frac{kc}{\omega_{\rm T}}\right)^2}{\left(\epsilon_r^{(0)} + \left(\frac{kc}{\omega_{\rm T}}\right)^2\right)^2}\right\}\right], \qquad \frac{kc}{\omega_{\rm T}} \to \infty\,. \tag{5.55}$$

As $kc/\omega_{\rm T} \to \infty$, we have

$$\left.\frac{\omega^2}{\omega_T^2}\right|_{\text{longt.}} \approx \frac{\epsilon_r^{(0)} + \left(\frac{kc}{\omega_{\rm T}}\right)^2}{2\,\epsilon^{(\infty)}} \times 2 \to \frac{\epsilon_r^{(0)}}{\epsilon_r^{(\infty)}} + \frac{1}{\epsilon_r^{(\infty)}} \left(\frac{kc}{\omega_{\rm T}}\right)^2 \approx \left(\frac{kc}{\sqrt{\epsilon_r^{(\infty)}}\,\omega_{\rm T}}\right)^2, \qquad \frac{kc}{\omega_{\rm T}} \to \infty\,, \tag{5.56}$$

$$\left.\frac{\omega^2}{\omega_{\rm T}^2}\right|_{\text{trans.}} \approx \frac{1}{2} \, \frac{4\epsilon_r^{(\infty)} \left(\frac{kc}{\omega_T}\right)^2}{2\epsilon_r^{(\infty)} \left(\epsilon_r^{(0)} + \left(\frac{kc}{\omega_{\rm T}}\right)^2\right)} = \frac{(kc/\omega_{\rm T})^2}{\epsilon_r^{(0)} + (kc/\omega_{\rm T})^2} \to 1\,, \qquad \frac{kc}{\omega_{\rm T}} \to \infty\,. \tag{5.57}$$

In two regions, the above dispersion relations are akin to electromagnetic waves. These are the longitudinal modes, for large k,

$$\left.\frac{\omega^2}{\omega_{\rm T}^2}\right|_{\text{longt.}} \approx \left(\frac{kc}{\sqrt{\epsilon_r^{(\infty)}}\,\omega_{\rm T}}\right)^2, \qquad k \to \infty\,, \tag{5.58}$$

and the transverse modes, for small k,

$$\left.\frac{\omega^2}{\omega_{\rm T}^2}\right|_{\text{trans.}} \approx \left(\frac{kc}{\sqrt{\epsilon_r^{(0)}}\,\omega_{\rm T}}\right)^2, \qquad k \to 0\,. \tag{5.59}$$

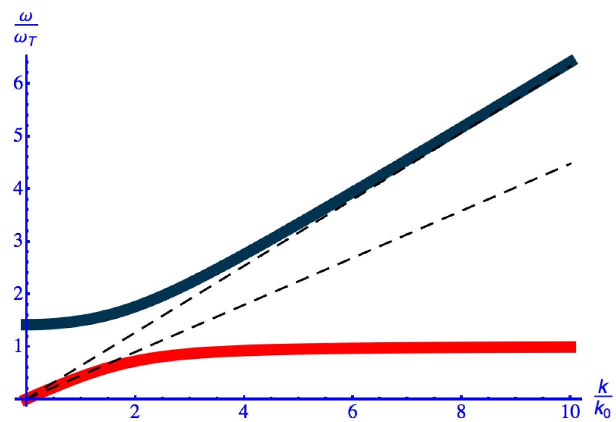


Figure 5.3: Illustration of the two branches of the electromagnetic spectrum of a typical solid. The values $\epsilon_r^{(0)} = 5.0$ and $\epsilon_r^{(\infty)} = 2.5$ are used. The quantity $k_0 = \omega_{\mathrm{T}}/c$ is a scaled wave vector.

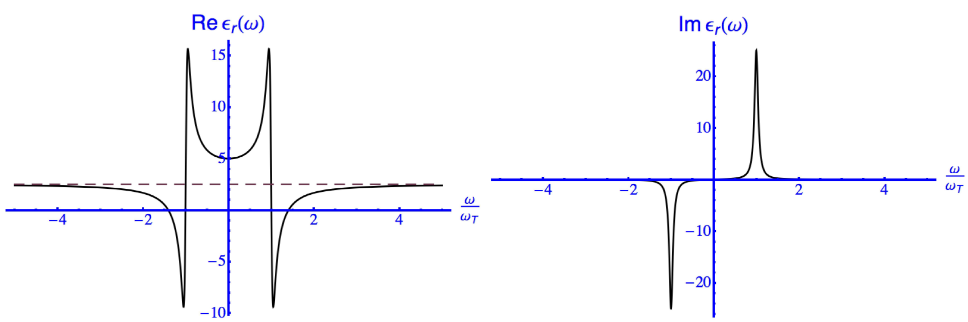


Figure 5.4: Illustration of the real and imaginary parts of $\epsilon_r(\omega)$, for the ionic crystal model. Again, the values $\epsilon_r^{(0)} = 5.0$ and $\epsilon_r^{(\infty)} = 2.5$ are used.

We have previously defined the longitudinal and transverse modes in the limit $k \to 0$. The above limits tell us that in the limit $k \to \infty$, the two modes actually interchange their role, the initially longitudinal mode becoming more transverse, and the initially transverse mode becoming more longitudinal. A truly transverse mode would fulfill a dispersion relation where ω is proportional to k. The relevant quantities entering the dispersion relations (5.58) and (5.59) are the limits $\epsilon_r^{(0)}$ and $\epsilon_r^{(\infty)}$ of the dielectric constant. This interplay is due to the optical polarizabilities and the crystal vibrations, as discussed above in Sec. 5.3.1.

5.3.4 Index of Refraction for an Ionic Crystal

Generally, when dealing with the propagation of an electromagnetic wave in a medium, the frequency of the wave is determined by the boundary conditions and sources. The relation between the wave vector and the angular frequency is given by $k = k_r + \mathrm{i}\, k_i$ with

$$k_r(\omega) = n(\omega)\,\frac{\omega}{c} = \mathrm{Re}\, k\,, \tag{5.60a}$$

$$k_i(\omega) = \kappa(\omega)\,\frac{\omega}{c} = \mathrm{Im}\, k\,. \tag{5.60b}$$

From the dispersion relation (5.44), one gets

$$\begin{aligned} k^2 = \epsilon_r(\omega)\,\frac{\omega^2}{c^2} &= [\mathrm{Re}\,\epsilon_r(\omega) + \mathrm{i}\,\mathrm{Im}\,\epsilon_r(\omega)]\,\frac{\omega^2}{c^2} \\ &= [n(\omega) + \mathrm{i}\,\kappa(\omega)]^2\,\frac{\omega^2}{c^2} \\ &= [n^2(\omega) - \kappa^2(\omega) + 2\,\mathrm{i}\,n(\omega)\,\kappa(\omega)]\,\frac{\omega^2}{c^2}\,. \end{aligned} \tag{5.61}$$

This can be summarized in the following equation,

Dispersion and Dielectric Constant: $$\epsilon_r(\omega) = \mathrm{Re}\,\epsilon_r(\omega) + \mathrm{i}\,\mathrm{Im}\,\epsilon_r(\omega) = [n(\omega) + \mathrm{i}\,\kappa(\omega)]^2\,, \tag{5.62}$$

or, neglecting the ambiguity introduced, as

$$\sqrt{\epsilon_r(\omega)} = n(\omega) + \mathrm{i}\,\kappa(\omega)\,. \tag{5.63}$$

We note that in all our models and calculation, $\epsilon_r(\omega)$ has a positive imaginary and a positive real part. We identity $\sqrt{\epsilon_r(\omega)} = \sqrt{|\epsilon_r(\omega)|}\,\exp[\frac{\mathrm{i}}{2}\arg(\epsilon_r(\omega))]$ with the "obvious" branch of the square root, consistent with the branch cut of the square root being along the negative real axis. Projected onto the real and imaginary parts, this means that

$$n^2(\omega) - \kappa^2(\omega) = \mathrm{Re}\,\epsilon_r(\omega)\,, \tag{5.64a}$$

$$2\,n(\omega)\,\kappa(\omega) = \mathrm{Im}\,\epsilon_r(\omega)\,, \tag{5.64b}$$

$$n^2(\omega) + \kappa^2(\omega) = |\epsilon(\omega)|\,. \tag{5.64c}$$

We can also deduce that

$$n^2(\omega) = \frac{1}{2}\{\mathrm{Re}\,\epsilon(\omega) + |\epsilon_r(\omega)|\}\,, \qquad n(\omega) > 0\,, \tag{5.65}$$
$$\kappa^2(\omega) = \frac{1}{2}\{-\mathrm{Re}\,\epsilon_r(\omega) + |\epsilon_r(\omega)|\}\,, \qquad \mathrm{sgn}\,\kappa(\omega) = \mathrm{sgn}\,\mathrm{Im}\,\epsilon_r(\omega)\,. \tag{5.66}$$

For an ionic crystal,

$$\begin{aligned} \epsilon_r(\omega) &= \epsilon_r^{(\infty)} + \omega_{\mathrm{T}}^2\,\frac{\epsilon_r^{(\infty)} - \epsilon_r^{(0)}}{\omega^2 + \mathrm{i}\,\gamma\,\omega - \omega_{\mathrm{T}}^2}\,\frac{\omega^2 - \mathrm{i}\,\gamma\,\omega - \omega_{\mathrm{T}}^2}{\omega^2 - \mathrm{i}\,\gamma\,\omega - \omega_{\mathrm{T}}^2} \\ &= \epsilon_r^{(\infty)} + \omega_{\mathrm{T}}^2\,\frac{\epsilon_r^{(\infty)} - \epsilon_r^{(0)}}{[\omega^2 - \omega_{\mathrm{T}}^2]^2 + \gamma^2\,\omega^2}\,\left(\omega^2 - \omega_{\mathrm{T}}^2 - \mathrm{i}\,\gamma\,\omega\right)\,. \end{aligned} \tag{5.67}$$

We can now gain a little more insight into the real and imaginary parts of the dilectric constant for an ionic crystal,

$$
\begin{aligned}
\mathrm{Re}\,\epsilon_r(\omega) &= \epsilon_r^{(\infty)} + \omega_{\mathrm{T}}^2 \, \frac{\epsilon_r^{(\infty)} - \epsilon_r^{(0)}}{\left(\omega^2 - \omega_{\mathrm{T}}^2\right)^2 + \gamma^2\,\omega^2} \, \left(\omega^2 - \omega_{\mathrm{T}}^2\right) \\
&= \epsilon_r^{(\infty)} + \omega_{\mathrm{T}}^2 \, \frac{\epsilon_r^{(\infty)} - \epsilon_r^{(\infty)}\,\omega_{\mathrm{L}}^2/\omega_{\mathrm{T}}^2}{\left(\omega^2 - \omega_{\mathrm{T}}^2\right)^2 + \gamma^2\,\omega^2} \left(\omega^2 - \omega_{\mathrm{T}}^2\right) \\
&= \epsilon_r^{(\infty)} \, \frac{\left(\omega^2 - \omega_{\mathrm{T}}^2\right)^2 + \gamma^2\,\omega^2 + \left(\omega_{\mathrm{T}}^2 - \omega_{\mathrm{L}}^2\right)\left(\omega^2 - \omega_{\mathrm{T}}^2\right)}{[\omega^2 - \omega_{\mathrm{T}}^2]^2 + \gamma^2\omega^2} \\
&= \epsilon_r^{(\infty)} \, \frac{\left(\omega^2 - \omega_{\mathrm{T}}^2\right)\,\left(\omega^2 - \omega_{\mathrm{T}}^2 + \omega_{\mathrm{T}}^2 - \omega_{\mathrm{L}}^2\right) + \gamma^2\,\omega^2}{\left(\omega^2 - \omega_{\mathrm{T}}^2\right)^2 + \gamma^2\,\omega^2} \\
&= \epsilon_r^{(\infty)} \, \frac{\left(\omega^2 - \omega_{\mathrm{T}}^2\right)\,\left(\omega^2 - \omega_{\mathrm{L}}^2\right) + \gamma^2\,\omega^2}{\left(\omega^2 - \omega_{\mathrm{T}}^2\right)^2 + \gamma^2\,\omega^2} \, .
\end{aligned}
\tag{5.68}
$$

The imaginary part can be written as

$$
\begin{aligned}
\mathrm{Im}\,\epsilon_r(\omega) &= -\,\omega_{\mathrm{T}}^2 \, \frac{\left(\epsilon_r^{(\infty)} - \epsilon_r^{(0)}\right)\,\gamma\,\omega}{\left(\omega^2 - \omega_{\mathrm{T}}^2\right)^2 + \gamma^2\,\omega^2} = \frac{\epsilon_r^{(\infty)}\,\left(\omega_{\mathrm{L}}^2 - \omega_{\mathrm{T}}^2\right)\,\gamma\,\omega}{\left(\omega^2 - \omega_{\mathrm{T}}^2\right)^2 + \gamma^2\,\omega^2} > 0\,, \\
|\epsilon(\omega)| &= \epsilon_r^{(\infty)} \, \frac{\left(\omega^2 - \omega_{\mathrm{L}}^2\right)^2 + \gamma^2\,\omega^2}{\left(\omega^2 - \omega_{\mathrm{T}}^2\right)^2 + \gamma^2\,\omega^2} \, .
\end{aligned}
\tag{5.69}
$$

The imaginary part is positive (we recall that $\omega_{\mathrm{L}} > \omega_{\mathrm{T}}$) and has a linear asymptotics for $\omega \to 0$, while it behaves as $1/\omega^3$ for $\omega \to \infty$. The electric field for a plane electromagnetic wave traveling in the x direction is given by

$$
\begin{aligned}
\vec{E}(x,t) &= \int_{-\infty}^{\infty} \vec{\mathcal{E}}(\omega)\,\exp\left[\mathrm{i}\,\left(k(\omega)\,x - \omega\,t\right)\right] \frac{\mathrm{d}\omega}{2\pi} \\
&= \int_{-\infty}^{\infty} \vec{\mathcal{E}}(\omega)\,\exp\left[\mathrm{i}\,\left(\frac{\omega}{c}\,\left(n(\omega) + \mathrm{i}\,\kappa\,(\omega)\right)\,x - \omega\,t\right)\right] \frac{\mathrm{d}\omega}{2\pi} \\
&= \int_{-\infty}^{\infty} \vec{\mathcal{E}}(\omega)\,\exp\left[\mathrm{i}\,\left(\frac{\omega}{c}\,n(\omega)\,x - \omega\,t\right)\right]\,\exp\left[-\frac{\omega}{c}\,\kappa(\omega)\,x\right] \frac{\mathrm{d}\omega}{2\pi} \, .
\end{aligned}
\tag{5.70}
$$

As seen from Fig. 5.4, in general, $\omega\,\kappa(\omega) \geq 0$ for all ω. For positive frequency $\omega > 0$, the planes of constant phase are found by

$$
\Delta\Phi = \frac{\omega}{c}\,n(\omega)\,\Delta x - \omega\,\Delta t = 0 \qquad \Rightarrow \qquad \frac{\Delta x}{\Delta t} = \frac{c}{n(\omega)} \, . \tag{5.71}
$$

Both positive and negative frequencies propagate into the $+x$ direction, and they are also damped in the

$+x$ direction. The solution under the interchange of $k(\omega) \to -k(\omega)$ also is a solution to the wave equation,

$$\begin{aligned}
\vec{E}\,(x,t) &= \int_{-\infty}^{\infty} \vec{\mathcal{E}}\,(\omega)\ \exp\left[-\mathrm{i}\ k(\omega)\ x - \omega\ t\right]\ \frac{\mathrm{d}\omega}{2\pi} \\
&= \int_{-\infty}^{\infty} \vec{\mathcal{E}}\,(\omega)\ \exp\left[-\mathrm{i}\ \left(\frac{\omega}{c}\ \left(n\,(\omega)\right) + \mathrm{i}\ \kappa\ (\omega)\right)\ x - \omega\ t\right]\ \frac{\mathrm{d}\omega}{2\pi} \\
&= \int_{-\infty}^{\infty} \vec{\mathcal{E}}\,(\omega)\ \exp\left[-\mathrm{i}\ \left(\frac{\omega}{c}\ n\,(\omega)\ \ x + \omega\ t\right)\right]\ \exp\left[\frac{\omega}{c}\ \kappa\,(\omega)\ \ x\right]\ \frac{\mathrm{d}\omega}{2\pi}\,.
\end{aligned} \tag{5.72}$$

Here, the planes of constant phase are found as

$$\Delta\Phi = -\frac{\omega}{c}\ n\,(\omega)\ \ \Delta x - \omega\ \Delta t = 0 \qquad \Rightarrow \qquad \frac{\Delta x}{\Delta t} = -\frac{c}{n\,(\omega)}\,. \tag{5.73}$$

In this case, both positive and negative frequencies propagate into the $-x$ direction. This is also the direction in which the wave is exponentially damped, in view of the factor

$$\exp\left[\frac{\omega}{c}\ \kappa\,(\omega)\ \ x\right] \to 0\,, \qquad x \to -\infty\,. \tag{5.74}$$

In both cases, the wave is exponentially damped in its propagation direction. One thus has to understand the variable k^2 in the relation (5.61) as $|\vec{k}|^2$, and determine the components of $\vec{k}$ according to the propagation direction.

5.3.5 Drude Model for the Dielectric Constant

Let us briefly review the main result for the dielectric response of a plasma, where electrons move about freely. This theory is relevant to the oscillations of an electron gas in a near-perfect conductor, where the "resonant excitation frequency" of the electrons is zero. Alternatively, a microscopic theory, using a statistical ansatz for the description of particles before and after the collisions (Boltzmann equation), leads to the result

$$\epsilon_r(\omega) = 1 - \frac{\sigma_0}{\epsilon_0}\ \frac{1}{(\omega + \mathrm{i}\eta)\ \ (\omega\,\tau_0 + \mathrm{i})} = 1 - \frac{\sigma_0}{\epsilon_0}\ \frac{1}{\omega\ \ (\omega\,\tau_0 + \mathrm{i})}\,. \tag{5.75}$$

Here, η is an infinitesimal parameter, σ_0 is a characteristic conductivity of the plasma, and τ_0 is the mean time between electron collisions in the plasma. This model is described by the general ansatz (5.1), if we set $\omega_{m=0} = 0$ and identify $\gamma_{m=0} = 1/\tau_0$. It is useful to collect a few results,

$$\epsilon(\omega) = \epsilon_0\ \epsilon_r(\omega)\,, \qquad \epsilon_r(\omega) = 1 + G(\omega)\,, \tag{5.76a}$$

$$\epsilon(\omega) = \epsilon_0\ [1 + G(\omega)]\,, \tag{5.76b}$$

$$\epsilon(t - t') = \epsilon_0\,\epsilon_r(t - t')\,, \tag{5.76c}$$

$$\epsilon(t - t') = \epsilon_0\,[\delta(t - t') + G(t - t')]\,, \tag{5.76d}$$

$$G(t - t') = \int \frac{\mathrm{d}\omega}{2\pi}\ \mathrm{e}^{-\mathrm{i}\omega(t-t')}\ G(\omega),\,. \tag{5.76e}$$

For a causal behaviour, $G(t-t')$ vanishes for $t-t'<0$. The dielectric displacement is obtained from the electric field as

$$\vec{D}(\vec{r},t) = \epsilon_0 \int_{-\infty}^{\infty} \delta(t-t')\,\vec{E}(\vec{r},t') + \epsilon_0 \int_{-\infty}^{\infty} \mathrm{d}t'\; G(t-t')\;\vec{E}(\vec{r},t')$$

$$= \epsilon_0\,\vec{E}(\vec{r},t) + \epsilon_0 \int_{0}^{\infty} \mathrm{d}\tau\; G(\tau)\;\vec{E}(\vec{r},t-\tau)\,, \tag{5.77}$$

where we use the fact that $G(t-t')=0$ for $t-t'<0$. This implies that $G(t-t')$ must have dimension of inverse time, or frequency. In frequency space, we have a simple multiplication,

$$\widetilde{\vec{D}}(\vec{r},\omega) = \epsilon_0\;\widetilde{\vec{E}}(\vec{r},\omega) + \epsilon_0\;\widetilde{G}(\vec{r},\omega)\;\widetilde{\vec{E}}(\vec{r},\omega)\,. \tag{5.78}$$

This implies that $\widetilde{G}(\vec{r},\omega)$ must have dimension of unity. The formulas

$$\widetilde{G}(\vec{r},\omega) = \int \mathrm{d}t\;\mathrm{e}^{\mathrm{i}\,\omega\,t}\,G(\vec{r},t)\,, \tag{5.79a}$$

$$G(\vec{r},t) = \int \frac{\mathrm{d}\omega}{2\pi}\;\mathrm{e}^{-\mathrm{i}\,\omega\,t}\;\widetilde{G}(\vec{r},\omega) \tag{5.79b}$$

confirm these different units for G and $\widetilde{G}$. Now, $\epsilon_r(\omega)$ has the same units as $\widetilde{G}$, and is thus dimensionless. The conductivity σ_0 relates the current flowing in the plasma to the electric field,

$$\vec{J} = \sigma_0\;\vec{E}\,. \tag{5.80}$$

Here, σ_0 has the dimension of $(\mathrm{C}/(\mathrm{m}^2\,\mathrm{s}))\,\mathrm{m}/\mathrm{V} = (\mathrm{C}/(\mathrm{m}\;\;\mathrm{s}))\,(\mathrm{C}/\mathrm{J}) = \mathrm{C}^2/(\mathrm{Jms})$. By contrast, ϵ_0 has dimension $\mathrm{A\,s}/(\mathrm{V\;\;m}) = \mathrm{C}^2/(\mathrm{J\;\;m})$. So, σ_0/ϵ_0 has dimension of frequency. So, the Drude ansatz for the dielectric constant, which we repeat for convenience,

$$\epsilon_r(\omega) = 1 - \frac{\sigma_0}{\epsilon_0}\;\frac{1}{\omega\;\;(\omega\,\tau_0 + \mathrm{i})} = 1 - \frac{\sigma_0}{\epsilon_0\,\tau_0}\;\frac{1}{\omega^2} + \mathcal{O}(\omega^{-3})\,, \qquad \omega\to\infty\,, \tag{5.81}$$

has the right physical dimension (unity).

Now, let us make contact with the usual identification of the current density. If we recall that according to Eq. (1.19),

$$\vec{J}_p\,(\vec{r},t) = \frac{\partial}{\partial t}\vec{P}\,(\vec{r},t)\;, \qquad\qquad \vec{J}_p\,(\vec{r},\omega) = -\mathrm{i}\,\omega\,\vec{P}\,(\vec{r},\omega)\;, \tag{5.82}$$

then we have

$$\vec{D}(\vec{r},\omega) = \epsilon_0\,\vec{E}(\vec{r},\omega) + \vec{P}(\vec{r},\omega) = \epsilon_r(\omega)\,\vec{E}(\vec{r},\omega) = \epsilon_0\,\vec{E}(\vec{r},\omega) - \sigma_0\;\frac{1}{\omega\;\;(\omega\,\tau_0+\mathrm{i})}\,\vec{E}(\vec{r},\omega)\,. \tag{5.83}$$

This allows us to identify

$$\vec{P}(\vec{r},\omega) = -\sigma_0\;\frac{1}{\omega\;\;(\omega\,\tau_0+\mathrm{i})}\,\vec{E}(\vec{r},\omega) \approx -\sigma_0\;\frac{1}{\mathrm{i}\omega}\,\vec{E}(\vec{r},\omega)\,. \tag{5.84}$$

Multiplying both sides by $-\mathrm{i}\omega$, we have

$$\vec{J}_p(\vec{r},\omega) = -\mathrm{i}\omega\vec{P}(\vec{r},\omega) \approx \sigma_0\;\vec{E}(\vec{r},\omega)\,, \tag{5.85}$$

which is fully consistent with the usual definition of the current density and conductivity σ_0 provided the total current $\vec{J}(\vec{r},\omega)$ is identified with the polarization current $\vec{J}_p(\vec{r},\omega)$.

Furthermore, matching the formula for $\epsilon_r(\omega)$ with the definition

$$\omega_p^2 \equiv \lim_{\omega\to\infty} \omega^2 \, [1 - \epsilon_r(\omega)] \tag{5.86}$$

for the plasma frequency leads to the equation

$$\omega_p^2 = \frac{\sigma_0}{\epsilon_0 \, \tau_0} \tag{5.87}$$

for the plasma frequency ω_p. On the other hand, using a microscopic theory of the plasma, we can establish that

$$\omega_p^2 = \frac{N_V \, e^2}{\epsilon_0 \, m_e} \,, \qquad\qquad \sigma_0 = \frac{N_V \, e^2}{m_e} \, \tau_0 \,. \tag{5.88}$$

where N_V is the volume density of free electrons and m_e is the electron mass. Let us calculate the Fourier backtransform of Eq. (5.75),

$$G(\omega) = -\frac{\sigma_0}{\epsilon_0} \, \frac{1}{\omega \, (\omega \, \tau_0 + \mathrm{i})} \,, \qquad G(t - t') = -\frac{\sigma_0}{\epsilon_0} \int \frac{\mathrm{d}\omega}{2\pi} \, \mathrm{e}^{-\mathrm{i}\,\omega\,(t-t')} \, \frac{1}{(\omega + \mathrm{i}\eta) \, (\omega \, \tau_0 + \mathrm{i})} \,, \tag{5.89}$$

where we introduce an infinitesimal imaginary part $\mathrm{i}\eta$ in order to ensure causality. So, the two residues are

$$\operatorname*{Res}_{\omega=-\mathrm{i}\eta} \, \mathrm{e}^{-\mathrm{i}\,\omega\,(t-t')} \, \frac{1}{(\omega + \mathrm{i}\eta) \, (\omega \, \tau_0 + \mathrm{i})} = \frac{1}{\mathrm{i}} \,, \tag{5.90a}$$

$$\operatorname*{Res}_{\omega=-\mathrm{i}/\tau_0} \, \mathrm{e}^{-\mathrm{i}\,\omega\,(t-t')} \, \frac{1}{(\omega + \mathrm{i}\eta) \, (\omega \, \tau_0 + \mathrm{i})} = \mathrm{e}^{-(t-t')/\tau_0} \left(-\frac{1}{\mathrm{i}} \right) \,. \tag{5.90b}$$

The prefactor multiplying the residues is $(-2\pi\mathrm{i})$, because the poles are encircled in the mathematically negative (clockwise) direction, for $t - t' > 0$. The end result is

$$G(t - t') = \Theta(t - t') \, \frac{\sigma_0}{\epsilon_0} \, \left[1 - \exp\left(-\frac{t - t'}{\tau_0} \right) \right] \,. \tag{5.91}$$

This quantity has dimension of frequency, as it should, because it is obtained as the Fourier backtransform of a dimensionless quantity $\epsilon_r(\omega)$. For perfect electrical conductors, $\tau_0 \to \infty$, and

$$G\,(t - t') \to \Theta(t - t') \, \frac{\sigma_0}{\epsilon_0} \,, \qquad\qquad \tau_0 \to \infty \,. \tag{5.92}$$

For a realistic conductor and finite τ_0, we have $G(t - t' = 0) = 0$, and the integral (5.77) converges.

It is an interesting exercise to relate the expression in Eq. (5.91) *to the Green function of the harmonic oscillator in the limit of vanishing resonant frequency* $\omega_0 \to 0$.

5.4 Propagation of Plane Waves in a Medium

5.4.1 Orientation

We now have investigated several models for the index of refraction. Our next task is to investigate the propagation of a plane electromagnetic wave in an ionic crystal and in a metal. Reviewing the permittivities obtained for the models, we note that the real part of the permittivity is an even function of ω, and the imaginary part is an odd function. The properties of the real and imaginary parts of the indices of refraction are obtained from those of the permittivities. Like the permittivity we find that the real part of the index of refraction is an even function of ω and the imaginary part is an odd function of ω,

$$n(\omega) = n(-\omega)\,, \qquad \kappa(\omega) = -\kappa(-\omega)\,. \tag{5.93}$$

Using these properties, a plane polarized wave traveling in the $+x$ direction can be described by the expression

$$\begin{aligned} \vec{E}\,(x,t) &= \int_0^\infty \vec{E}_0\,(\omega)\ \exp\left[\mathrm{i}\ \left(\mathrm{Re}\ k(\omega) + \mathrm{i}\ \mathrm{Im}\ k(\omega)\right)\ x - \omega t\right]\ \frac{\mathrm{d}\omega}{2\pi} + \mathrm{c.c.} \\ &= \int_0^\infty \vec{E}_0\,(\omega)\ \exp\left[\mathrm{i}\ \frac{\omega}{c}\,(n\,(\omega)\ x - c\,t)\right]\ \exp\left(-\frac{\omega}{c}\ \kappa\,(\omega)\ x\right)\ \frac{\mathrm{d}\omega}{2\pi} + \mathrm{c.c.}\,. \end{aligned} \tag{5.94}$$

We will restrict our analysis to the case in which $\left|\vec{E}_0\,(\omega)\right|$ has a single, 'narrow' peak of width $\Delta\omega$ at ω_0. The peak will be considered narrow if

$$\frac{\Delta\omega}{n\,(\omega_0)}\ \frac{\mathrm{d}n\,(\omega_0)}{\mathrm{d}\omega_0} \ll 1\,. \tag{5.95}$$

5.4.2 Group Velocity

For a wave composed of a narrow range of frequencies, the wave vector $\vec{k}\,(\omega)$ can be expanded about the central frequency to obtain an approximate expression for the propagation of the wave. In our example the central frequency is ω_0, and

$$\frac{\omega}{c}\ n\,(\omega) = \frac{\omega_0}{c} n\,(\omega_0) + \left[n\,(\omega_0) + \omega_0\ \frac{\mathrm{d}n\,(\omega_0)}{\mathrm{d}\omega_0}\right]\ \frac{\omega - \omega_0}{c} + \frac{1}{2}\ \left[2\ \frac{\mathrm{d}n\,(\omega_0)}{\mathrm{d}\omega_0} + \omega_0\ \frac{\mathrm{d}^2 n\,(\omega_0)}{\mathrm{d}\omega_0^2}\right]\ \frac{(\omega-\omega_0)^2}{c} + \ldots \tag{5.96}$$

While a similar expression holds for $\omega\kappa\,(\omega)\,/c$, we here set $\kappa(\omega) = \kappa(\omega_0)$. Working to first order in $\omega - \omega_0$ we find

$$\begin{aligned} \vec{E}\,(\vec{r},t) = {} & \exp\left(\mathrm{i}\frac{\omega_0}{c}\ \left(n\,(\omega_0)\,x - c\,t\right)\right)\ \exp\left(-\frac{\omega_0}{c}\ \kappa\,(\omega_0)\ x\right) \\ & \times \int_0^\infty \vec{E}_0\,(\omega) \exp\left(\mathrm{i}\ \frac{\omega - \omega_0}{c}\ \left\{\left[n\,(\omega_0) + \omega_0\ \frac{\mathrm{d}n\,(\omega_0)}{\mathrm{d}\omega_0}\right]\ x - c\,t\right\}\right)\ \frac{\mathrm{d}\omega}{2\pi} + \mathrm{c.c.}\,, \end{aligned} \tag{5.97}$$

where $\mathrm{c.c.}$ stands for complex conjugation. The interpretation is as follows: The first factor identifies a term that oscillates at frequency ω_0 and propagates at the phase velocity $c/n(\omega_0)$. The second term is an integral (envelope function) that travels with the group velocity

$$v_g\,(\omega_0) = c\ \left[n\,(\omega_0) + \omega_0\ \frac{\mathrm{d}n\,(\omega_0)}{\mathrm{d}\omega_0}\right]^{-1}\,, \tag{5.98}$$

i.e.

$$\begin{aligned} \vec{E}\,(\vec{r},t) = {} & \underbrace{\exp\left(\mathrm{i}\frac{\omega_0}{c}\ \left(n\,(\omega_0)\,x - c\,t\right)\right)\ \exp\left(-\frac{\omega_0}{c}\ \kappa\,(\omega_0)\ x\right)}_{\text{damped wave traveling at phase velocity}} \\ & \times \underbrace{\int_0^\infty \vec{E}_0\,(\omega) \exp\left(\mathrm{i}\ \frac{\omega - \omega_0}{v_g(\omega_0)}\ \left\{x - v_g(\omega_0)\,t\right\}\right)\ \frac{\mathrm{d}\omega}{2\pi}}_{\text{wave traveling at group velocity}} + \mathrm{c.c.} \\ = {} & \exp\left(\mathrm{i}\frac{\omega_0}{c}\ \left(n\,(\omega_0)\,x - c\,t\right)\right)\ \exp\left(-\frac{\omega_0}{c}\ \kappa\,(\omega_0)\ x\right)\ \vec{F}(x - v_g\,(\omega_0)\ t)\,. \end{aligned} \tag{5.99}$$

This should describe a 'broad' wave packet with the envelope function traveling with speed $v_g\,(\omega_0)$ and the underlying wave traveling with speed $c/n\,(\omega_0)$.

In order to establish the relation to the usual definition of the group velocity, we observe that

$$\mathrm{Re}\,k = n(\omega)\,\frac{\omega}{c}\,, \qquad \left.\frac{\mathrm{dRe}\,k}{\mathrm{d}\omega}\right|_{\omega=\omega_0} = \frac{1}{c}\left(n(\omega_0) + \omega_0\,\left.\frac{\mathrm{d}n(\omega)}{\mathrm{d}\omega}\right|_{\omega=\omega_0}\right) = \frac{1}{v_g}\,. \tag{5.100}$$

The usual definition of the group velocity reads as $v_g = \mathrm{d}\omega/\mathrm{d}k \to \mathrm{d}\omega/\mathrm{d}(\mathrm{Re}\,k)$. We have learned that k can be complex, hence the modification. Furthermore, it is actually possible to turn a differential quotient upside down, so the definitions are consistent.

5.4.3 Causality and Contour Integration

In the Fourier backtransformation of the dielectric constant of a plasma, we have already seen that in order to ensure causality, we have to displace all poles in ω integrations infinitesimally below the real axis. Here, we study an example where this general notion is exemplified in a particularly clear and general way.

Let $u\,(x,t)$ be a wave propagating in a medium with a permittivity $\epsilon_r\,(\omega)$ and let

$$\boxed{\textbf{temporarily:}} \qquad n\,(\omega) \equiv \sqrt{\epsilon_r\,(\omega)}\,. \tag{5.101}$$

We temporarily change the notation here slightly, in order to unify $n(\omega)$ and $\kappa(\omega)$ into a single symbol. Then, a signal $u(x,t)$ can be written as the Fourier backtransform of a sum of waves, propagating in the $+x$ and/or $-x$ directions, as follows,

$$u\,(x,t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\left[A\,(\omega)\,\exp\left(\mathrm{i}\frac{\omega\,n\,(\omega)\,x}{c}\right) + B\,(\omega)\,\exp\left(-\mathrm{i}\frac{\omega\,n\,(\omega)\,x}{c}\right)\right]\,. \tag{5.102}$$

Let $u\,(0,t)$ and $\partial u\,(x,t)\,/\partial x|_{x=0}$ be given. Then, in particular,

$$u\,(0,t) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t}\,\left[A\,(\omega) + B\,(\omega)\right]\frac{\mathrm{d}\omega}{2\pi}\,, \tag{5.103a}$$

$$\begin{aligned}\Rightarrow \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,\omega'\,t}\,u\,(0,t)\,\mathrm{d}t &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(\omega-\omega')t}\,\left[A\,(\omega) + B\,(\omega)\right]\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \left[A\,(\omega) + B\,(\omega)\right]\,\delta(\omega-\omega')\,\mathrm{d}\omega = A\,(\omega') + B\,(\omega')\,, \end{aligned} \tag{5.103b}$$

and similarly

$$\left.\frac{\partial}{\partial x}u\,(x,t)\right|_{x=0} = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t}\left[\mathrm{i}\frac{\omega\,n\,(\omega)}{c}\,A\,(\omega) - \mathrm{i}\frac{\omega\,n\,(\omega)}{c}\,B\,(\omega)\right]\,\frac{\mathrm{d}\omega}{2\pi}\,, \tag{5.104a}$$

$$\begin{aligned}\Rightarrow \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega' t}\,\left.\frac{\partial}{\partial x}u\,(x,t)\right|_{x=0}\,\mathrm{d}t &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{i}\frac{\omega n\,(\omega)}{c}\,\mathrm{e}^{-\mathrm{i}(\omega-\omega')t}\,\left[A\,(\omega) - B\,(\omega)\right]\,\frac{\mathrm{d}\omega}{2\pi}\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \left[A\,(\omega) - B\,(\omega)\right]\,\mathrm{i}\frac{\omega\,n\,(\omega)}{c}\,\delta(\omega-\omega')\,\mathrm{d}\omega\,,\end{aligned}$$

$$-\mathrm{i}\,\frac{c}{\omega'\,n\,(\omega')}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega' t}\,\left.\frac{\partial}{\partial x}u\,(x,t)\right|_{x=0}\,\mathrm{d}t = A\,(\omega') - B\,(\omega')\,. \tag{5.104b}$$

Solving Eqs. (5.103b) and (5.104b) for $A(\omega')$ and $B(\omega')$, and setting

$$\partial_x u(x,t) \equiv \frac{\partial}{\partial x} u(x,t) \,, \qquad\qquad \partial_x u(0,t) \equiv \left. \frac{\partial}{\partial x} u(x,t) \right|_{x=0} \,, \tag{5.105}$$

one obtains

$$A(\omega') = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega' t} \left\{ u(0,t) - \mathrm{i} \frac{c}{\omega' \, n(\omega')} \, \partial_x u(0,t) \right\} \mathrm{d}t \,, \tag{5.106a}$$

$$B(\omega') = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega' t} \left\{ u(0,t) + \mathrm{i} \frac{c}{\omega' \, n(\omega')} \, \partial_x u(0,t) \right\} \mathrm{d}t \,. \tag{5.106b}$$

The Fourier expansion coefficients $A(\omega')$ and $B(\omega')$ can thus be reconstructed from integrals over the signal and its derivative, evaluated at $x = 0$ but observed over all time, i.e., from $-\infty$ to $+\infty$.

Under the interchange $\omega \leftrightarrow \omega'$ and $t \leftrightarrow t'$, these formulas can be used directly in Eq. (5.102),

$$\begin{aligned} u(x,t) = {} & \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}t' \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \, \mathrm{e}^{-\mathrm{i}\omega\,(t-t')} \, \exp\left(\mathrm{i} \frac{\omega\, n(\omega)\, x}{c} \right) \left\{ u(0,t') - \frac{\mathrm{i}c}{\omega\, n(\omega)} \, \partial_x u(0,t') \right\} \\ & + \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}t' \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \, \mathrm{e}^{-\mathrm{i}\omega\,(t-t')} \, \exp\left(-\mathrm{i} \frac{\omega\, n(\omega)\, x}{c} \right) \left\{ u(0,t') + \frac{\mathrm{i}c}{\omega\, n(\omega)} \, \partial_x u(0,t') \right\} \,. \end{aligned} \tag{5.107}$$

One might ask the following question: Under the integral signs on the right-hand side, we integrate over the entire range of t', i.e., $t' \in (-\infty, \infty)$. However, on the left-hand side, we have $u(x,t)$. The suspicion would be that the signal at $t' > t$ as it enters the integrand on the right-hand side might influence the left-hand side in a non-causal way. In order to ensure causality, our only chance is to slightly deform the integration over ω into a contour integration in the complex ω plane into a contour C, which is conveniently chosen to lie infinitesimally above the real axis, just like for the retarded Green function. We thus write

$$\begin{aligned} u(x,t) = {} & \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}t' \int_{C} \frac{\mathrm{d}\omega}{2\pi} \, \mathrm{e}^{-\mathrm{i}\,\omega\,(t-t')} \, \exp\left(\mathrm{i} \frac{\omega\, n(\omega)\, x}{c} \right) \left\{ u(0,t') - \frac{\mathrm{i}c}{\omega\, n(\omega)} \, \partial_x u(0,t') \right\} \\ & + \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}t' \int_{C} \frac{\mathrm{d}\omega}{2\pi} \, \mathrm{e}^{-\mathrm{i}\,\omega\,(t-t')} \, \exp\left(-\mathrm{i} \frac{\omega\, n(\omega)\, x}{c} \right) \left\{ u(0,t') + \frac{\mathrm{i}c}{\omega\, n(\omega)} \, \partial_x u(0,t') \right\} \,. \end{aligned} \tag{5.108}$$

In order to observe the constraints imposed by causality, it is sufficient to consider the following expressions for $x > 0$,

$$g(x, t-t') = -\frac{1}{2} \int_{C} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega(t-t')} \, \exp\left(\mathrm{i} \frac{\omega\, n(\omega)\, x}{c} \right) \, \frac{\mathrm{i}c}{\omega n(\omega)} \,, \tag{5.109a}$$

$$\partial_x g(x, t-t') = \frac{1}{2} \int_{C} \frac{\mathrm{d}\omega}{2\pi} \, \mathrm{e}^{-\mathrm{i}\omega(t-t')} \, \exp\left(\mathrm{i} \frac{\omega\, n(\omega)\, x}{c} \right) \,, \tag{5.109b}$$

$$h(x, t-t') = \frac{1}{2} \int_{C} \frac{\mathrm{d}\omega}{2\pi} \, \mathrm{e}^{-\mathrm{i}\omega(t-t')} \, \exp\left(-\mathrm{i} \frac{\omega\, n(\omega)\, x}{c} \right) \, \frac{\mathrm{i}c}{\omega n(\omega)} \,, \tag{5.109c}$$

$$\partial_x h(x, t-t') = \frac{1}{2} \int_{C} \frac{\mathrm{d}\omega}{2\pi} \, \mathrm{e}^{-\mathrm{i}\omega(t-t')} \, \exp\left(-\mathrm{i} \frac{\omega\, n(\omega)\, x}{c} \right) \,. \tag{5.109d}$$

Equation (5.108) can thus be written as follows,

$$\begin{aligned} u(x,t) = {} & \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}t' \, \left\{ \partial_x g(x, t-t') \; u(0,t') + g(x, t-t') \; \partial_x u(0,t') \right\} \\ & + \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}t' \, \left\{ \partial_x h(x, t-t') \; u(0,t') + h(x, t-t') \; \partial_x u(0,t') \right\} \,. \end{aligned} \tag{5.110}$$

A general property of all expressions for $\epsilon_r(\omega)$ and $n(\omega)$ discussed so far is that for large $|\omega|$, they approach unity. In order to obtain well behaved integrands for large $|\omega|$, we investigate this asymptotic dependence. First, we consider the functions obtained with the vacuum refractive index $n(\omega) = 1$, and call them $g_0(x, t-t')$ and $\partial_x g_0(x, t-t')$. It is very instructive to evaluate them using a contour in the lower half plane and the Cauchy residue theorem. The residue is that of a simple pole at $\omega = 0$, and since we close the contour in the lower half plane, it is a full pole (not a half pole),

$$\begin{aligned} g_0(x, t-t') = & -\frac{1}{2}\int_C \mathrm{e}^{-\mathrm{i}\omega\left(t-t'\right)}\,\exp\left(\mathrm{i}\frac{\omega x}{c}\right)\,\frac{\mathrm{i}\,c}{\omega}\frac{\mathrm{d}\omega}{2\pi} = -\frac{1}{2}\left(-2\pi\mathrm{i}\right)\frac{1}{2\pi}\left(\mathrm{i}c\right)\Theta\left(t-t'-\frac{x}{c}\right) \\ = & -\frac{c}{2}\,\Theta\left(t-t'-\frac{x}{c}\right), \end{aligned} \tag{5.111a}$$

$$\partial_x g_0(x, t-t') = \frac{1}{2}\int_C \mathrm{e}^{-\mathrm{i}\omega\left(t-t'\right)}\,\exp\left(\mathrm{i}\frac{\omega x}{c}\right)\mathrm{d}\omega = \frac{1}{2}\delta\left(t-t'-\frac{x}{c}\right). \tag{5.111b}$$

As expected, $g_0(x, t-t')$ and $f_0(x, t-t')$ are "causal" in that a disturbance at the origin at time t' does not affect the wave at x until a point in time t, which satisfies $c(t-t') = x$. We now consider $g(x, t-t') - g_0(x, t-t')$, where $g(x, t-t')$ is formulated for a nonvanishing dispersion $n(\omega) \neq 1$. We write this as

$$g(x, t-t') - g_0(x, t-t') = -\frac{\mathrm{i}\,c}{2}\int_C \frac{\mathrm{e}^{-\mathrm{i}\omega\left(t-t'-x/c\right)}}{\omega}\left\{\frac{\exp\left(\mathrm{i}\omega\left[n(\omega)-1\right]x/c\right)}{n(\omega)} - 1\right\}\frac{\mathrm{d}\omega}{2\pi}. \tag{5.112}$$

This is the Fourier transform of a function that vanishes as $\omega \to \pm\infty$, because $n(\pm\infty) = 1$. For $x > c(t-t')$, we have

$$t - t' - x/c < 0, \qquad x > c(t-t'). \tag{5.113}$$

The function $g(x, t-t') - g_0(x, t-t')$ is zero for $x > c(t-t')$ if

$$\Gamma(\omega) \equiv \frac{1}{\omega}\left\{\frac{\exp\left(\mathrm{i}\omega\left[n(\omega)-1\right]x/c\right)}{n(\omega)} - 1\right\} \tag{5.114}$$

has no singularities or cuts in the upper half complex ω plane. The requirement that $\Gamma(\omega)$ have no cuts or poles in the upper half complex ω plane is satisfied if $n(\omega) = \sqrt{\epsilon_r(\omega)}$ has no zeros or cuts in the upper half complex plane. In this case, $\epsilon_r(\omega)$ can neither have a pole nor a cut in the upper half plane. This condition is satisfied by the model permittivities for the ionic crystals and for the charged plasmas.

In general, if

$$\epsilon_r(\omega) = \operatorname{Re}\epsilon_r(\omega) + \mathrm{i}\operatorname{Im}\epsilon_r(\omega), \qquad \operatorname{Re}\epsilon_r(\omega) > 0, \qquad \operatorname{sgn}[\operatorname{Im}\epsilon_r(\omega)] = \operatorname{sgn}(\omega), \tag{5.115}$$

then

$$\operatorname{Im}\left(\frac{1}{\omega\sqrt{\epsilon_r(\omega)}}\right) = \operatorname{Im}\left(\frac{1}{\omega\, n(\omega)}\right) < 0. \tag{5.116}$$

We then have

$$g(x, t-t') - g_0(x, t-t') = 0, \qquad x > c(t-t'). \tag{5.117}$$

A similar argument applies to $\partial_x g(x, t-t') - \partial_x g_0(x, t-t')$. This function can be written as

$$\partial_x g(x, t-t') - \partial_x g_0(x, t-t') = \frac{1}{2}\int_C \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega\left(t-t'-x/c\right)}\left[\exp\left(\mathrm{i}\frac{\omega\left[n(\omega)-1\right]x/c}{c}\right) - 1\right]. \tag{5.118}$$

Again, the function being transformed vanishes as $\omega \to \pm\infty$ provided $\epsilon_r(\omega)$ has no poles or zeros in the upper half plane. So,

$$\partial_x g(x, t-t') - \partial_x g_0(x, t-t') = 0, \qquad x > c(t-t'). \tag{5.119}$$

The analysis can be repeated for the functions h and $\partial_x h$ defined in Eqs. (5.109c) and (5.109d).

5.4.4 Steepest Descent Method for Wave Dispersion in a Medium

The method of steepest descent is a powerful method used for the evaluation of a number of integrals relevant to physics. Here, we observe that, surprisingly, the method of steepest descent also has an application in the propagation of wave packets in electrodynamics.

In order to examine the dispersion character for a wave in a medium, we take

$$u(0,t) = u_0\,\delta(t)\,, \qquad \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0\,. \tag{5.120}$$

These boundary conditions will provide identical waves traveling in the $+x$ and $-x$ directions. Since the Fourier transform of a delta function is a constant in frequency space, this acts as a "white light" source. The generated signal is, by virtue of Eq. (5.108),

$$\begin{aligned} u(x,t) &= \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}t\int_C \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}\left[\exp\left(\mathrm{i}\frac{\omega\,n(\omega)\,x}{c}\right)+\exp\left(-\mathrm{i}\frac{\omega\,n(\omega)\,x}{c}\right)\right]u(0,t')\;\mathrm{d}\omega\,\mathrm{d}t' \\ &= u_0\int_C \frac{\mathrm{d}\omega}{2\pi}\,e^{-\mathrm{i}\omega t}\cos\left(\frac{\omega n(\omega)\,x}{c}\right)\,. \end{aligned} \tag{5.121}$$

Since $u(x,t) = u(-x,t)$ it suffices to consider the wave traveling in either the $+x$ or $-x$ direction,

$$\begin{aligned} u(x,t) &= \frac{1}{2}\,u_0\int_C \frac{\mathrm{d}\omega}{2\pi}\;\exp\left(-\mathrm{i}\omega\left[t-\frac{n(\omega)\,x}{c}\right]\right)+\frac{1}{2}\,u_0\int_C \frac{\mathrm{d}\omega}{2\pi}\;\exp\left(-\mathrm{i}\omega\left[t+\frac{n(\omega)\,x}{c}\right]\right) \\ &= u_+(x,t)+u_-(x,t)\;. \end{aligned} \tag{5.122}$$

We investigate the wave traveling in the $+x$ direction,

$$\begin{aligned} u_+(x,t) &= \frac{1}{2}\,u_0\int_C \frac{\mathrm{d}\omega}{2\pi}\;\exp\left(-\mathrm{i}\omega\left[t-\frac{n(\omega)\,x}{c}\right]\right)\,, \\ u_+^*(x,t) &= \frac{1}{2}\,u_0\int_C \frac{\mathrm{d}\omega}{2\pi}\;\exp\left(\mathrm{i}\omega\left[t-\frac{n^*(\omega)\,x}{c}\right]\right)\,. \end{aligned} \tag{5.123}$$

Since $n(\omega) = n^*(-\omega)$, and ignoring the distortion of the contour to a place infinitesimally above the real axis, the above integrals extend from $\omega = -\infty$ to $\omega = +\infty$ and are thus equal. That is to say, with $-\infty,\ldots,\infty$ being replaced by $0,\ldots,\infty$, we have

$$u_+(x,t) = u_0\;\mathrm{Re}\int_0^{\infty}\frac{\mathrm{d}\omega}{2\pi}\;\exp\left(-\mathrm{i}\omega\left[t-\frac{n(\omega)\,x}{c}\right]\right)\,. \tag{5.124}$$

This integral will be the object of our desire for the time being. In order to explore the usefulness of the steepest descent method, we now approximate

$$n(\omega) \approx \left(1-\frac{\omega_p^2}{\omega^2}\right)^{1/2}\,, \tag{5.125}$$

so that

$$\omega\,n(\omega) = \sqrt{\omega^2-\omega_p^2}\,. \tag{5.126}$$

The method of steepest descent is a technique for obtaining an approximation to an integral of the form

$$\int_C \exp\left[\alpha\,u(t)\right]\mathrm{d}t \tag{5.127}$$

for large α. The geometric location of the saddle points of $u(z)$, for complex z, is found, and the path of integration is distorted so as to pass through these saddle points. In going through a saddle point, the path is along the line for which the curvature is negative. In the vicinity of a saddle point, one may interpret the function

$$u = u(z) = u(\mathrm{Re}\, z, \mathrm{Im}\, z)\,, \tag{5.128}$$

as a function of two variables (real and imaginary part). The magnitude of $\exp[\alpha\, u(z)]$ is determined by the real part of $u(z)$. Along the path of steepest descent, one heads down to the valley of $\mathrm{Re}(u)$ as soon as possible. This may imply that one has to distort the path of integration in the complex plane, in order to "hit" the saddle point at the "right" angle. Of course, if the integration variable is real and the saddle point is obtained by going straight through the saddle point along the real axis, then there is no need for a distortion into the complex plane. The basic idea is that any maximum or minimum of $u(z)$, given by a saddle point, can be turned into a maximum of $\exp[\alpha u(t)]$ if the path in the complex z plane is rotated appropriately. Furthermore, the idea of the method of steepest descent is that the Gaussian integral obtained by the expansion of the argument function $u(z)$ about the saddle point, can actually be evaluated with ease. As $\mathrm{i}\, u(z)$ can be regarded as the phase of the integrand, the method is also known as the "stationary phase approximation." The classic reference for this is Courant & Hilbert, Methods of Mathematical Physics, Vol. 1, Interscience Publishers, pp. 526-532.

After this detour, let us return to the evaluation of

$$u_+(x,t) = u_0\, \mathrm{Re} \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \exp\left(-\mathrm{i}\omega \left[t - \frac{n(\omega)\, x}{c}\right]\right)\,. \tag{5.129}$$

by the method of steepest descent. The application of the steepest descent method is facilitated in our case by the fact that

$$\omega\, n(\omega) = \begin{cases} \sqrt{\omega^2 - \omega_p^2}\,, & \omega > \omega_p \\ \\ \mathrm{i}\sqrt{\omega_p^2 - \omega^2}\,, & \omega_p > \omega \end{cases}\,. \tag{5.130}$$

With these observations, the function can be written as

$$\begin{aligned} u_+(x,t) = {} & \frac{u_0}{2\pi}\, \mathrm{Re} \int_{\omega_p}^\infty \exp\left(-\mathrm{i}\,\omega\, \left[t - \sqrt{\omega^2 - \omega_p^2}\, \frac{x}{c}\right]\right) \mathrm{d}\omega \\ & + \frac{u_0}{2\pi}\, \mathrm{Re} \int_0^{\omega_p} \exp\left(-\omega\, \left[\mathrm{i}t + \sqrt{\omega_p^2 - \omega^2}\, \frac{x}{c}\right]\right) \mathrm{d}\omega \end{aligned} \tag{5.131}$$

Our problem is to obtain an estimate for the value of this function if $\omega_p\,(x/c) \gg 1$. The second integral is exponentially decaying with distance and so will be assumed to be negligible. If we insist on the condition $\omega_p\,(x/c) \gg 1$, then, given the range of typical plasma frequencies, from $10^9\ \mathrm{rad/s}$ for gaseous plasmas through $10^{16}\ \mathrm{rad/s}$ for metallic plasmas (conducting electrons in a metal), the distance of the observation point from the source should be much greater than $10\,\mathrm{cm}$ for the gaseous plasmas and $1\,\mu\mathrm{m}$ for the metals.

We thus estimate

$$u_+(x,t) \approx \frac{u_0}{2\pi}\, \mathrm{Re} \int_{\omega_p}^\infty \exp\left(-\mathrm{i}\,\omega\, \left[t - \sqrt{\omega^2 - \omega_p^2}\, \frac{x}{c}\right]\right) \mathrm{d}\omega = \frac{u_0}{2\pi}\, \mathrm{Re} \int_{\omega_p}^\infty \mathrm{e}^{-\mathrm{i}\,\phi(\omega)}\, \mathrm{d}\omega \tag{5.132}$$

by the method of steepest descent. We have defined the complex phase of the integrand as

$$\phi(\omega) = \omega\, \left[t - \sqrt{\omega^2 - \omega_p^2}\, \frac{x}{c}\right] \tag{5.133}$$

Regions in which the phase varies "rapidly" will contribute little to the integral. For large $\omega_p\, t_0$, the major contribution to the integral will come from the region where the magnitude of the phase is minimum or has zero slope. This occurs at the critical value $\omega = \omega_c$, where

$$\frac{\partial \phi}{\partial \omega} = 0\,, \qquad \omega = \omega_c = \frac{\omega_p\, c\, t}{\sqrt{(c\,t)^2 - x^2}}\,. \qquad \phi(\omega_c) = \sqrt{(c\,t)^2 - x^2}\;\; \frac{\omega_p}{c}\,, \tag{5.134}$$

$$\phi''(\omega_c) = \left.\frac{\partial^2 \phi}{\partial \omega^2}\right|_{\omega=\omega_c} = \frac{\left((ct)^2 - x^2\right)^{3/2}}{c\,x^2\,\omega_p}\,, \qquad \phi(\omega) \approx \phi(\omega_c) + \frac{1}{2}\,\phi''(\omega_c)\,(\omega - \omega_c)^2\,. \tag{5.135}$$

We have assumed $ct > x$ in writing these expressions. The method of steepest descent thus approximates

$$u_+\,(x,t) \approx \frac{u_0}{2\pi}\,\mathrm{Re}\int_{\omega_p}^{\infty} \mathrm{e}^{-\mathrm{i}\,\phi(\omega)}\;\mathrm{d}\omega \approx \frac{u_0}{2\pi}\,\mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}\,\phi(\omega_c)}\int_{\omega_p}^{\infty} \mathrm{e}^{-\mathrm{i}\frac{1}{2}\,\phi''(\omega_c)\,(\omega-\omega_c)^2}\;\mathrm{d}\omega\right] \tag{5.136}$$

The decisive step of the method of steepest descent is to replace in this expression

$$J = \int_{\omega_p}^{\infty} \mathrm{e}^{-\mathrm{i}\,\frac{1}{2}\,\phi''(\omega_c)\,(\omega-\omega_c)^2}\;\mathrm{d}\omega \to \int_C \mathrm{e}^{-\mathrm{i}\,\frac{1}{2}\,\phi''(\omega_c)\,(\omega-\omega_c)^2}\;\mathrm{d}\omega\,, \tag{5.137}$$

where C is a contour of steepest descent of the magnitude of the integrand, and to extend this contour to infinity, the idea being that the integrand should decrease sufficiently rapidly in order to ensure that this extension is possible. If we parameterize

$$\omega - \omega_c = \omega(w) = \exp\left(-\mathrm{i}\frac{\pi}{4}\right)\, w = \left(\frac{1}{\sqrt{2}} - \frac{\mathrm{i}}{\sqrt{2}}\right)\, w\,, \tag{5.138}$$

then we can extend the contour C onto the interval $w \in (-\infty, \infty)$ and have

$$\begin{aligned} J &= \exp\left(-\mathrm{i}\frac{\pi}{4}\right)\int_C \mathrm{e}^{-\frac{1}{2}\phi''(\omega_c)\,w^2}\;\mathrm{d}w \\ &\to \exp\left(-\mathrm{i}\frac{\pi}{4}\right)\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}\,\phi''(\omega_c)\,w^2}\;\mathrm{d}w = \mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{\frac{2\pi}{\phi''(\omega_c)}}\,. \end{aligned} \tag{5.139}$$

Hence,

$$\begin{aligned} u_+\,(x,t) &\approx \frac{u_0}{2\pi}\,\mathrm{Re}\left\{\exp\left[-\mathrm{i}\,\left(\phi(\omega_c) + \frac{\pi}{4}\right)\right]\sqrt{\frac{2\pi}{\phi''(\omega_c)}}\right\} = \frac{u_0}{2\pi}\sqrt{\frac{2\pi}{\phi''(\omega_c)}}\,\cos\left(\phi(\omega_c) + \frac{\pi}{4}\right) \\ &= \frac{u_0}{2\pi}\sqrt{2\pi}\left(\frac{\left((ct)^2 - x^2\right)^{3/2}}{c\,x^2\,\omega_p}\right)^{-1/2} \cos\left(\sqrt{(c\,t)^2 - x^2}\;\;\frac{\omega_p}{c} + \frac{\pi}{4}\right) \\ &= \frac{u_0}{\sqrt{2\pi}}\,\frac{\sqrt{c}\,\sqrt{\omega_p}\,x}{\left((ct)^2 - x^2\right)^{3/4}}\cos\left(\frac{\sqrt{(c\,t)^2 - x^2}}{c}\omega_p + \frac{\pi}{4}\right)\,. \end{aligned} \tag{5.140}$$

This result fulfills the condition that $u(x=0,t) = u_0\,\delta(t) = 0$ for $t > 0$, in that it vanishes at $x = 0$.

In the region $x > c\,t$, the value of the critical phase $f(\omega_c)$ becomes imaginary, and the contribution of the saddle point is exponentially suppressed. Furthermore, an evaluation of the variation about the saddle point (along the contour of steepest descent, which in the latter case is parallel to the imaginary axis) then yields the wisdom that the saddle point contribution happens to be purely imaginary (for $x > c\,t$).

Hence, the result given in Eq. (5.140) for the "causal" region $c\,t > x$ is the complete result obtained by the steepest descent method.

It is an interesting problem to derive a relation analogous to Eq. (5.140) *for the quantity* $u_-(x,t)$.

5.5 Kramers–Kronig Relationships

5.5.1 Analyticity and the Kramers–Kronig Relationship

We recall once more the basic relations relevant for the dielectric constant and the electric field,

$$\epsilon(\omega) = \epsilon_0\, \epsilon_r(\omega)\,, \tag{5.141}$$

$$\epsilon(\omega) = \epsilon_0\, [1 + G(\omega)]\,, \tag{5.142}$$

$$\epsilon(t - t') = \epsilon(\tau) = \epsilon_0\, \epsilon_r(\tau)\,, \tag{5.143}$$

$$\epsilon(t - t') = \epsilon(\tau) = \epsilon_0\, [1 + G(\tau)]\,. \tag{5.144}$$

For a causal behaviour, $G(\tau)$ vanishes for $\tau < 0$. The following is the convolution that calculates $G(\tau)$,

$$\vec{D}(\vec{r},t) = \epsilon_0\, \vec{E}(\vec{r},t) + \epsilon_0 \int_0^\infty \mathrm{d}\tau\, G(\tau)\, \vec{E}(\vec{r},t-\tau)\,, \tag{5.145}$$

and in frequency space,

$$\tilde{\vec{D}}(\vec{r},\omega) = \epsilon_0\, \tilde{\vec{E}}(\vec{r},\omega) + \epsilon_0\, \widetilde{G}(\omega)\, \tilde{\vec{E}}(\vec{r},\omega)\,. \tag{5.146}$$

The harmonic oscillator displacement $x(t)$ (polarization) at a given time is expressed by the Green function in terms of the force (electric field). Likewise, for the polarization, the displacement is the dipole moment $qx(t)$, and the Green function dependence is calculated with respect to the driving force, which in this case is given by the applied, external electric field (the Green function then is the atomic polarizability). We have determined that causality requires $\epsilon(\omega)$ to be analytic and to have no zeros in the upper half complex ω plane. In the case that the local approximation holds, this can be seen by considering the causal expression

$$\vec{D}\,(\vec{r},t) = \epsilon_0\, \vec{E}\,(\vec{r},t) + \epsilon_0 \int_0^\infty G\,(\tau)\;\vec{E}\,(\vec{r},t-\tau)\;\mathrm{d}\tau\,. \tag{5.147}$$

In this expression, $G\,(\tau)$ is a real rather than complex function; it represents a property of the system and is generally expected to vanish as $\tau \to \infty$, so that the integral converges at the upper limit. The Fourier transform, in time, of this expression along with the relationship between the frequency components of $\vec{D}$ and $\vec{E}$ yields

$$\int_{-\infty}^\infty \vec{D}\,(\vec{r},t)\,\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t = \epsilon_0 \int_{-\infty}^\infty \vec{E}\,(\vec{r},t)\;\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t + \epsilon_0 \int_{-\infty}^\infty \int_0^\infty G\,(\tau)\,\vec{E}\,(\vec{r},t-\tau)\;\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t\,\mathrm{d}\tau\,, \tag{5.148a}$$

$$\vec{D}\,(\vec{r},\omega) = \epsilon_0\, \vec{E}\,(\vec{r},\omega) + \epsilon_0 \int_{-\infty}^\infty \int_0^\infty G\,(\tau)\,\mathrm{e}^{\mathrm{i}\omega\tau}\;\vec{E}\,(\vec{r},t-\tau)\;\mathrm{e}^{\mathrm{i}\omega(t-\tau)}\mathrm{d}(t-\tau)\,\mathrm{d}\tau\,, \tag{5.148b}$$

$$\epsilon_r\,(\omega)\;\epsilon_0\,\tilde{\vec{E}}\,(\vec{r},\omega) = \epsilon_0\,\tilde{\vec{E}}\,(\vec{r},\omega) + \epsilon_0 \int_0^\infty G\,(\tau)\,\mathrm{e}^{\mathrm{i}\omega\tau}\;\tilde{\vec{E}}\,(\vec{r},\omega)\,\mathrm{d}\tau\,. \tag{5.148c}$$

Division by $\tilde{\vec{E}}(\vec{r},\omega)$ yields the relation

$$\epsilon_r(\omega) - 1 = \int_0^\infty G(\tau)\ \exp(\mathrm{i}\omega\tau)\ \mathrm{d}\tau\,. \tag{5.149}$$

Since $\vec{D}$ and $\vec{E}$ are real, $G(\tau)$ will also be real. It follows, as previously noted, that

$$\epsilon_r(\omega) = \epsilon_r(-\omega)^*\ . \tag{5.150}$$

It is also possible to obtain a relationship between G, its derivatives at $\tau = 0$, and the high frequency behavior of $\epsilon_r(\omega)$. This relationship can be obtained by integration by parts. To arrive at the relationship, we first note that

$$\int_0^\infty G(\tau)\ \exp(\mathrm{i}\omega\tau)\ \mathrm{d}\tau = \frac{1}{\mathrm{i}\omega}\left(\left[G(\tau)\ \exp(\mathrm{i}\omega\tau)\right]\Big|_{\tau=0^+}^{\tau=\infty} - \int_0^\infty \frac{\mathrm{d}G(\tau)}{\mathrm{d}\tau}\ \exp(\mathrm{i}\omega\tau)\ \mathrm{d}\tau\right)\,. \tag{5.151}$$

Assuming damping at $\tau\to\infty$, it makes sense to re-apply integration by parts and to obtain

$$\int_0^\infty \frac{\mathrm{d}^n G(\tau)}{\mathrm{d}\tau^n}\ \exp(\mathrm{i}\omega\tau)\ \mathrm{d}\tau = \frac{1}{\mathrm{i}\omega}\ \left(\left[\frac{\mathrm{d}^n G(\tau)}{\mathrm{d}\tau^n}\exp(\mathrm{i}\omega\tau)\right]\Big|_{\tau=0^+}^{\tau=\infty} - \int_0^\infty \frac{\mathrm{d}^{n+1} G(\tau)}{\mathrm{d}\tau^{n+1}}\ \exp(\mathrm{i}\omega\tau)\,\mathrm{d}\tau\right)\,. \tag{5.152}$$

For all materials which are not DC conductors, the derivatives in the infinite future are exponentially suppressed,

$$\lim_{n\to\infty}\frac{\mathrm{d}^n G(\tau)}{\mathrm{d}\tau^n} = 0\,, \qquad \tau\to\infty\,. \tag{5.153}$$

Counting the inverse powers of ω, induced by the partial integrations, we have

$$\epsilon_r(\omega) - 1 = \sum_{n=1}^{\infty}\frac{(-1)^n}{(\mathrm{i}\,\omega)^n}\ \frac{\mathrm{d}^{n-1}G(\tau')}{\mathrm{d}\tau'^{n-1}}\Big|_{\tau'=0^+}\,. \tag{5.154}$$

This expression helps to determine the asymptotic behavior of $\epsilon_r(\omega)$ for large ω. It is reasonable to expect that the response of the system takes a finite time, i.e., the system does not instantaneously change. In the case that this is true, we have that $G(0) = 0$. This is equivalent to the following observation. The first term for large ω, for $G(0)\neq 0$, would be created by the first term on the left-hand side of Eq. (5.151) and read $-G(0)/(\mathrm{i}\omega)$. However, a term proportional to $1/\omega$ would not be integrable in the Fourier-backtransformation to coordinate space (time). The first two terms in the expansion thus are

$$\epsilon_r(\omega) - 1 = -\frac{G'(0)}{\omega^2} - \mathrm{i}\,\frac{G''(0)}{\omega^3} + \dots\,, \tag{5.155a}$$

$$\mathrm{Re}(\epsilon_r(\omega) - 1) \approx -\frac{G'(0)}{\omega^2}\,, \tag{5.155b}$$

$$\mathrm{Im}(\epsilon_r(\omega) - 1) \approx -\frac{G''(0)}{\omega^3} + \dots\,. \tag{5.155c}$$

That is, the real part of $\epsilon(\omega) - 1$ vanishes as ω^{-2} and the imaginary part of $\epsilon(\omega)$ vanishes as ω^{-3}. Finally, for complex ω in the upper half plane, $\mathrm{Im}\,\omega > 0$, we have

$$\epsilon_r(\omega) - 1 = \int_0^\infty G(\tau)\ \mathrm{e}^{\mathrm{i}\omega\tau}\mathrm{d}\tau \le \int_0^\infty G(\tau)\ \left|\mathrm{e}^{\mathrm{i}\tau\mathrm{Re}\,\omega}\right|\ \mathrm{e}^{-\tau\mathrm{Im}\,\omega}\ \mathrm{d}\tau \le \int_0^\infty G(\tau)\,\mathrm{d}\tau\,, \qquad \mathrm{Im}\,\omega > 0\,. \tag{5.156}$$

If $\int_0^\infty G(\tau)\,d\tau$ is finite then $\epsilon(\omega) - 1$ will be finite and an analytic function of complex ω in the upper half of the complex plane, i.e., for $\mathrm{Im}\,\omega > 0$. This has some important consequences. The function $\tilde{\epsilon}(\omega) - 1$ exists for real ω and is the limit of a function which is analytic in the upper half plane.

5.5.2 Real and Imaginary Parts of the Kramers–Kronig Relation

In view of its analyticity requirements, one can derive important relations that interconnect the real and imaginary parts of the dielectric constant, as a function of frequency. These are called the Kramers-Kronig relations.

Since $\epsilon_r(\omega) - 1$ is an analytic function in the upper half complex plane then, by Cauchy's residue theorem,

$$\epsilon_r(z) = 1 + \frac{1}{2\pi\mathrm{i}} \oint_C \frac{\epsilon_r(z') - 1}{z' - z}\,\mathrm{d}z'\,, \qquad \mathrm{Im}\, z > 0\,, \tag{5.157}$$

with the imaginary part of all points on the curve C restricted to be greater than or equal to zero. Let the path lie along the real axis and let it be closed by the semi-circle of infinite radius. We assume that $\epsilon(\omega) - 1$ will vanish on this large semi-circle [consult Eqs. (5.155b) and (5.155c)]. In addition, let $z = \omega + \mathrm{i}\delta$, with $0 < \delta \ll 1$. Then,

$$\epsilon_r(\omega) = 1 + \frac{1}{2\pi\mathrm{i}} \lim_{\delta\to 0} \int_{-\infty}^{\infty} \frac{\epsilon_r(\omega') - 1}{\omega' - (\omega + \mathrm{i}\delta)}\,\mathrm{d}\omega' \tag{5.158}$$

Drawing the integration contour, it is immediately obvious that the integral taken below the pole equals the principal part of the integral plus $\pi\mathrm{i}$ times the residue at $\omega' = \omega$. The following identity is known as the Sokhotskii–Plemelj prescription,

$$\frac{1}{\omega' - \omega - \mathrm{i}\delta} = (\mathrm{P.V.})\,\frac{1}{\omega' - \omega} + \mathrm{i}\pi\,\delta(\omega' - \omega)\,. \tag{5.159}$$

It can be shown, based on considering the variable ω' as the independent variable. One observes that the pole is at $\omega' = \omega + \mathrm{i}\delta$. Therefore, under the integral sign, one would encircle the pole from below. Then, one adds and subtracts one half times the contour integral, with the contour being completed atop as opposed to below the pole. When one adds the other contour integral, one obtains the principle value; when on subtracts the upper contour integral, one completes the path around the pole, in the positive direction, leading to the term $\mathrm{i}\pi\,\delta(\omega' - \omega)$.

Therefore,

$$\begin{aligned}\epsilon_r(\omega) &= 1 + \frac{1}{2\pi\mathrm{i}} \left((\mathrm{P.V.}) \int_{-\infty}^{\infty} \frac{\epsilon_r(\omega') - 1}{\omega' - \omega}\,\mathrm{d}\omega' + \mathrm{i}\pi\,(\epsilon_r(\omega) - 1) \right) \\ &= 1 + \frac{1}{2\pi\mathrm{i}}\,(\mathrm{P.V.}) \int_{-\infty}^{\infty} \frac{\epsilon_r(\omega') - 1}{\omega' - \omega}\,\mathrm{d}\omega' + \frac{1}{2}\,(\epsilon_r(\omega) - 1) \\ &= \frac{1}{2}\epsilon_r(\omega) + \frac{1}{2} + \frac{1}{2\pi\mathrm{i}}\,(\mathrm{P.V.}) \int_{-\infty}^{\infty} \frac{\epsilon_r(\omega') - 1}{\omega' - \omega}\,. \end{aligned} \tag{5.160}$$

Projecting this equation onto real and imaginary parts, and observing that $\epsilon_r(\omega')$ has both a real as well as an imaginary part, one obtains

$$\mathrm{Re}\,\epsilon_r(\omega) = 1 + \frac{1}{\pi}\,(\mathrm{P.V.}) \int_{-\infty}^{\infty} \frac{\mathrm{Im}\,\epsilon_r(\omega')}{\omega' - \omega}\mathrm{d}\omega'\,, \tag{5.161a}$$

$$\mathrm{Im}\,\epsilon_r(\omega) = -\frac{1}{\pi}(\mathrm{P.V.}) \int_{-\infty}^{\infty} \frac{\mathrm{Re}\,\epsilon_r(\omega') - 1}{\omega' - \omega}\,\mathrm{d}\omega'\,. \tag{5.161b}$$

Using $\mathrm{Re}\,\epsilon_r(\omega) = \mathrm{Re}\,\epsilon_r(-\omega)$ and $\mathrm{Im}\,\epsilon_r(\omega) = -\mathrm{Im}\,\epsilon_r(-\omega)$, this can be written as

$$\begin{aligned}
\mathrm{Re}\,\epsilon_r(\omega) &= 1 + \frac{1}{\pi}\,(\mathrm{P.V.})\left[\int_{-\infty}^{0} \frac{\mathrm{Im}\,\epsilon_r(\omega')}{\omega'-\omega}\,\mathrm{d}\omega' + \int_0^{\infty} \frac{\mathrm{Im}\,\epsilon_r(\omega')}{\omega'-\omega}\,\mathrm{d}\omega'\right] \\
&= 1 + \frac{1}{\pi}\,(\mathrm{P.V.})\left[\int_{0}^{\infty} \frac{\mathrm{Im}\,\epsilon_r(-\omega')}{-\omega'-\omega}\,\mathrm{d}\omega' + \int_0^{\infty} \frac{\mathrm{Im}\,\epsilon_r(\omega')}{\omega'-\omega}\,\mathrm{d}\omega'\right] \\
&= 1 + \frac{1}{\pi}\,(\mathrm{P.V.})\left[\int_{0}^{\infty} \frac{\mathrm{Im}\,\epsilon_r(\omega')}{\omega'+\omega}\,\mathrm{d}\omega' + \int_0^{\infty} \frac{\mathrm{Im}\,\epsilon_r(\omega')}{\omega'-\omega}\,\mathrm{d}\omega'\right] \\
&= 1 + \frac{2}{\pi}\,(\mathrm{P.V.})\int_0^{\infty} \frac{\omega'\,\mathrm{Im}\,\epsilon_r(\omega')}{\omega'^2-\omega^2}\,\mathrm{d}\omega'\,, \qquad (5.162a)
\end{aligned}$$

for the real part, and for the imaginary part, we have

$$\begin{aligned}
\mathrm{Im}\,\epsilon_r(\omega) &= -\frac{1}{\pi}\,(\mathrm{P.V.})\left[\int_{-\infty}^{0} \frac{\mathrm{Re}\,\epsilon_r(\omega')-1}{\omega'-\omega}\,\mathrm{d}\omega' + \int_0^{\infty} \frac{\mathrm{Re}\,\epsilon_r(\omega')-1}{\omega'-\omega}\,\mathrm{d}\omega'\right] \\
&= -\frac{1}{\pi}\,(\mathrm{P.V.})\left[\int_{0}^{\infty} \frac{\mathrm{Re}\,\epsilon_r(-\omega')-1}{-\omega'-\omega}\,\mathrm{d}\omega' + \int_0^{\infty} \frac{\mathrm{Re}\,\epsilon_r(\omega')-1}{\omega'-\omega}\,\mathrm{d}\omega'\right] \\
&= -\frac{1}{\pi}\,(\mathrm{P.V.})\left[-\int_{0}^{\infty} \frac{\mathrm{Re}\,\epsilon_r(\omega')-1}{\omega'+\omega}\,\mathrm{d}\omega' + \int_0^{\infty} \frac{\mathrm{Re}\,\epsilon_r(\omega')-1}{\omega'-\omega}\,\mathrm{d}\omega'\right] \\
&= -\frac{2\omega}{\pi}\,(\mathrm{P.V.})\int_0^{\infty} \frac{\mathrm{Re}\,\epsilon_r(\omega')-1}{\omega'^2-\omega^2}\,\mathrm{d}\omega'\,. \qquad (5.162b)
\end{aligned}$$

These are the classic Kramers-Kronig relations between the real and imaginary parts of the electric permittivity. Similar relations hold for the frequency response function for any causal system. Similar relationships are very useful in quantum mechanics (particularly in scattering theory), circuit analysis in electrical engineering, etc.

The significance of these relations is that it provides a way to measure or calculate one characteristic of a system and then determine a completely different characteristic property of the system. For example, the existence of an absorption peak can be seen to lead to anomalous dispersion, i.e., $\mathrm{d}n(\omega)/\mathrm{d}\omega < 0$. This is easily demonstrated. Assume that the system has an absorption peak at $\omega = \omega_0$. Let us assume that

$$\epsilon_r(\omega) = 1 + \frac{\alpha}{\omega_0 - \omega - \mathrm{i}\,\frac{1}{2}\,\gamma} + \frac{\alpha}{\omega_0 + \omega + \mathrm{i}\,\frac{1}{2}\,\gamma} \approx 1 + \frac{2\alpha\omega_0}{\omega_0^2 - \omega^2 - \mathrm{i}\,\gamma\,\omega}\,, \qquad (5.163a)$$

$$\frac{1}{\omega_0 - \omega - \mathrm{i}\delta} = (\mathrm{P.V.})\,\frac{1}{\omega_0 - \omega} + \mathrm{i}\pi\,\delta(\omega_0 - \omega)\,, \qquad (5.163b)$$

$$\frac{\gamma}{2} \to \delta \ll 1\,, \qquad (5.163c)$$

$$\mathrm{Im}\,\epsilon_r(\omega) \approx \mathrm{i}\pi\,\alpha\,\omega_0\,\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]\,. \qquad (5.163d)$$

For simplicity, we thus take

$$\mathrm{Im}\,\epsilon_r(\omega) = \pi\,\alpha\,\omega_0\,\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]\,. \qquad (5.164)$$

where the additional terms vary slowly in the region of ω_0, and α is a dimensionless constant. The Kramers–Kronig relation (5.162) gives the real part of $\epsilon_r(\omega)$ in a region near ω_0 (but not at ω_0),

$$
\begin{aligned}
\mathrm{Re}\,\epsilon_r(\omega) &\approx 1 + \frac{2}{\pi}(\mathrm{P.V.})\int_0^\infty \frac{\omega'}{\omega'^2-\omega^2}\ (\pi\,\alpha\,\omega_0)\ [\delta(\omega'-\omega_0)-\delta(\omega'+\omega_0)]\ \mathrm{d}\omega' \\
&= 1+\frac{2\,\alpha\,\omega_0^2}{\omega_0^2-\omega^2}+\text{zero}\,, \qquad \text{(only the delta function } \delta(\omega'-\omega_0) \text{ contributes)}\,. \qquad (5.165)
\end{aligned}
$$

The contribution of the absorption peak to the real part of $\epsilon_r(\omega)$ is seen as a rapid variation in $\mathrm{Re}\,\epsilon(\omega)$ from positive for $\omega \lesssim \omega_0$ to negative for $\omega \gtrsim \omega_0$. The real part of $\epsilon_r(\omega)$ is an even function of ω, while the imaginary part is an odd function of ω.

It is also possible to use known values of $\epsilon_r(\omega)$ to aid with the calculation of the unknown values. For example, suppose we have measured or calculated the imaginary part of the index of refraction (i.e., the absorption for the system) and the real part of $\epsilon_r(\omega)$ is known, but only for $\omega=\omega_1$. Then, from Eq. (5.162), $\epsilon_r(\omega_1)$ satisfies

$$\mathrm{Re}\,\epsilon_r(\omega_1) = 1+\frac{2}{\pi}\ (\mathrm{P.V.})\int_0^\infty \frac{\omega'\ \mathrm{Im}\ \epsilon_r(\omega')}{\omega'^2-\omega_1^2}\ \mathrm{d}\omega'\,. \qquad (5.166)$$

It follows that

$$
\begin{aligned}
\mathrm{Re}\,\epsilon_r(\omega) - \mathrm{Re}\,\epsilon_r(\omega_1) &= \frac{2}{\pi}\ (\mathrm{P.V.})\int_0^\infty \omega'\mathrm{Im}\ \epsilon_r(\omega')\left(\frac{1}{\omega'^2-\omega^2}-\frac{1}{\omega'^2-\omega_1^2}\right)\ \mathrm{d}\omega' \\
&= \frac{2}{\pi}\ \left(\omega^2-\omega_1^2\right)\ (\mathrm{P.V.})\int_0^\infty \frac{\omega'\ \mathrm{Im}\ \epsilon_r(\omega')}{(\omega'^2-\omega^2)\ (\omega'^2-\omega_1^2)}\,\mathrm{d}\omega'\,. \qquad (5.167)
\end{aligned}
$$

This form for $\epsilon_r(\omega)$ reduces the integral's dependence on the values of $\mathrm{Im}\epsilon(\omega')$ at large ω', because of the ω'^{-4} dependence for large ω'. This is known as a "subtracted dispersion relation." This process of subtraction can be repeated for each known value of $\epsilon_r(\omega)$. A similar relationship can be generated for the imaginary part $\epsilon_r(\omega)$.

5.5.3 Sum Rules

The Kramers–Kronig relations can be used to obtain some general properties of the integrals of the real and imaginary parts of the permittivity. We have noted that, at high frequency, the permittivity varies as $\omega^{-2}+\mathcal{O}\left(\omega^{-3}\right)$. Physically, at high frequencies the binding forces for the electrons become negligible and the system responds as a plasma. With this interpretation, we define the plasma frequency for the system as

$$\omega_p^2 \equiv \lim_{\omega\to\infty}\left[\omega^2\ \left(1-\epsilon_r(\omega)\right)\right]\ . \qquad (5.168)$$

This definition of the plasma frequency reflects the contributions of the responses of all charged particles in the system. The coefficients in Sellmeier's equation, which approximates the permittivity as a sum of resonances, gives the "strength" of each resonance. In principle the sources of the resonances include optically active phonons, plasmons, electronic excitations, electronic ionizations, and others.

For a functional form of the kind introduced in Eq. (5.1),

$$\tilde{\epsilon}_r(\omega) = 1+\sum_m \frac{\mathcal{A}_m}{\omega_m^2-\omega^2-\mathrm{i}\gamma_m\omega}\,, \qquad (5.169)$$

one has

$$\omega_p^2 = \sum_m \mathcal{A}_m \tag{5.170}$$

by direct inspection of the limit. Using the Kramers–Kronig relation, it is possible to derive a more universal integral representation of ω_p^2 as defined in Eq. (5.168). Namely, if $\mathrm{Im}\ \epsilon_r(\omega)$ varies as $\omega^{-3} + \mathcal{O}\left(\omega^{-4}\right)$ for high frequencies, then Eq. (5.162) gives

$$\begin{aligned}
\omega_p^2 &= \lim_{\omega\to\infty}\left[\omega^2\left(1 - \mathrm{Re}\ \epsilon_r(\omega)\right) - \omega^2 \mathrm{i}\ \mathrm{Im}\ \epsilon_r(\omega)\right] = \lim_{\omega\to\infty}\left[\omega^2\left(1 - \mathrm{Re}\ \epsilon_r(\omega)\right)\right] \\
&= \lim_{\omega\to\infty}\left[-\omega^2\,\frac{2}{\pi}\,(\mathrm{P.V.})\int_0^\infty \frac{\omega'\ \mathrm{Im}\ \epsilon_r(\omega')}{\omega'^2 - \omega^2}\mathrm{d}\omega'\right] \\
&= \lim_{\omega\to\infty}\left[-\frac{2}{\pi}\ (\mathrm{P.V.})\ \int_0^\infty \underbrace{\frac{1}{\left[\omega'^2/\omega^2 - 1\right]}}_{\to -1}\ \omega'\ \mathrm{Im}\epsilon_r(\omega')\ \mathrm{d}\omega'\right] \\
&= \frac{2}{\pi}(\mathrm{P.V.})\ \int_0^\infty \omega\ \mathrm{Im}\ \epsilon_r(\omega)\ \mathrm{d}\omega = \frac{2}{\pi}\int_0^\infty \omega\ \mathrm{Im}\ \epsilon_r(\omega)\ \mathrm{d}\omega\,.
\end{aligned} \tag{5.171}$$

This is the sum rule for *oscillator strengths*. If the imaginary part itself has no singularities along the integration contour, we can dispose of the principal value prescription, as done in the last step.

For real ω, the imaginary part of the dielectric constant is given by Eq. (5.3),

$$\mathrm{Im}\ \epsilon_r(\omega) = \sum_m \frac{\omega\ \gamma_m\ \mathcal{A}_m}{\left(\omega^2 - \omega_m^2\right)^2 + \omega^2\ \gamma_m^2}\,. \tag{5.172}$$

It is interesting to consider the relationship between the coefficients in Sellmeier's equation, Eq. (5.1) , for the permittivity and the plasma frequency defined in Eq. (5.171).The sum rule requires that

$$\begin{aligned}
\omega_p^2 &= \frac{2}{\pi}\ \sum_m\ \int_0^\infty \frac{\omega^2\ \gamma_m\ \mathcal{A}_m}{(\omega_m^2 - \omega^2)^2 + (\gamma_m\omega)^2}\ \mathrm{d}\omega \\
&= \frac{1}{\pi}\ \sum_m\ \int_{-\infty}^\infty \frac{\omega^2\ \gamma_m\ \mathcal{A}_m}{\left(\omega_m^2 - \omega^2\right)^2 + (\gamma_m\omega)^2}\ \mathrm{d}\omega\,.
\end{aligned} \tag{5.173}$$

The imaginary part is a real function only if the argument ω is real. When ω itself becomes imaginary, then the "imaginary part" itself acquires an imaginary part. It is then possible to continue the imaginary part as a function into the complex plane. The poles are then the poles of a meromorphic function, originally defined to coincide with the imaginary part of the dilelectric function for real argument ω. This integrand has four poles, at

$$\begin{aligned}
\omega_1 &= \sqrt{\omega_m^2 - \frac{\gamma_m^2}{4}} + \mathrm{i}\frac{\gamma_m}{2}\,, \qquad \omega_2 = \sqrt{\omega_m^2 - \frac{\gamma_m^2}{4}} - \mathrm{i}\frac{\gamma_m}{2}\,, \\
\omega_3 &= -\sqrt{\omega_m^2 - \frac{\gamma_m^2}{4}} - \mathrm{i}\frac{\gamma_m}{2}\,, \qquad \omega_4 = -\sqrt{\omega_m^2 - \frac{\gamma_m^2}{4}} + \mathrm{i}\frac{\gamma_m}{2}\,,
\end{aligned} \tag{5.174}$$

In this case, we rewrite the integral as

$$\omega_p^2 = \frac{1}{\pi}\ \sum_m\ \int_{-\infty}^\infty \frac{\omega^2\ \gamma_m\ \mathcal{A}_m}{(\omega - \omega_1)\ (\omega - \omega_2)\ (\omega - \omega_3)\ (\omega - \omega_4)}\ \mathrm{d}\omega \tag{5.175}$$

and evaluate it using Cauchy's residue theorem. Closing the path in the upper half complex ω plane, we obtain the residues of the poles in the upper half plane (at ω_1 and ω_4)

$$\begin{aligned}\omega_p^2 &= 2\mathrm{i}\sum_m \frac{\omega_1^2\gamma_m\,\mathcal{A}_m}{(\omega_1-\omega_2)\;(\omega_1-\omega_3)\;(\omega_1-\omega_4)} + 2\mathrm{i}\sum_m \frac{\omega_4^2\gamma_m\,\mathcal{A}_m}{(\omega_4-\omega_1)\;(\omega_4-\omega_2)\;(\omega_4-\omega_3)} \\ &= -2\mathrm{i}\sum_m \frac{\omega_2^2\gamma_m\,\mathcal{A}_m}{(\omega_2-\omega_1)\;(\omega_2-\omega_3)\;(\omega_2-\omega_4)} - 2\mathrm{i}\sum_m \frac{\omega_3^2\gamma_m\,\mathcal{A}_m}{(\omega_3-\omega_1)\;(\omega_3-\omega_2)\;(\omega_3-\omega_4)} = \sum_m \mathcal{A}_m\,.\end{aligned} \tag{5.176}$$

The first version is obtained by closing the contour in the upper half complex plane, the second version is obtained by closing the contour in the lower half complex plane. It is an easy exercise to verify that both closings of the contour give consistent results. When closing the contour in the lower half plane, one has to consider the poles at ω_2 and ω_3. This is left as an exercise to the interested reader.

One can derive other sum rules. Let us assume that the real part of the dielectric function goes as $\mathrm{Re}\;\epsilon_r(\omega) - 1 \sim -\omega_p^2/\omega^2 + \mathcal{O}\left(\omega^{-4}\right)$ for large ω whereas the imaginary part goes as $\mathrm{Im}\;\epsilon_r(\omega) \sim \mathcal{O}\left(\omega^{-3}\right)$ for large ω. The asymptotic relationship between the real and imaginary parts of $\epsilon(\omega)$ can now be used to take advantage of the asymptotic properties in the range of large ω, say, $\omega \gg \omega_L > \omega_p$. We write the Kramers–Kronig relation as

$$\mathrm{Im}\;\epsilon_r(\omega) = \frac{2}{\pi\omega}\left[(\mathrm{P.V.})\int_0^{\omega_L} \frac{\mathrm{Re}\;[\epsilon_r(\omega')-1]}{1-(\omega'/\omega)^2}\mathrm{d}\omega' + \int_{\omega_L}^{\infty} \frac{\mathrm{Re}\;[\epsilon_r(\omega')-1]}{1-(\omega'/\omega)^2}\mathrm{d}\omega'\right]. \tag{5.177}$$

Again, we assume that ω_L is sufficiently large that the asymptotic form for the real part of $\epsilon(\omega)$ provides a valid approximation. The approximated equation is

$$\begin{aligned}\mathrm{Im}\;\epsilon_r(\omega) &\approx \frac{2}{\pi\omega}(\mathrm{P.V.})\int_0^{\omega_L} \frac{\mathrm{Re}\;[\epsilon_r(\omega')-1]}{1-(\omega'/\omega)^2}\mathrm{d}\omega' \\ &\quad + \frac{2}{\pi\omega}\,(\mathrm{P.V.})\int_{\omega_L}^{\infty} \frac{1}{1-(\omega'/\omega)^2}\left[-\frac{\omega_p^2}{\omega'^2} + \mathcal{O}\left(\omega'^{-4}\right)\right]\mathrm{d}\omega' \\ &\approx \frac{2}{\pi\omega}\left[(\mathrm{P.V.})\int_0^{\omega_L} \mathrm{Re}\;[\epsilon_r(\omega')-1]\,\mathrm{d}\omega' - \frac{\omega_p^2}{\omega_L} + \mathcal{O}\left(\omega^{-2}\right)\right].\end{aligned} \tag{5.178}$$

On the one hand, we have derived an approximation for the coefficent of order ω^{-1} of $\mathrm{Im}\;\epsilon_r(\omega)$, for large ω. On the other hand, since by assumption $\mathrm{Im}\;\epsilon_r(\omega) \sim \mathcal{O}\left(\omega^{-3}\right)$ for large ω, it follows that the expression just derived must vanish,

$$(\mathrm{P.V.})\int_0^{\omega_L} \mathrm{Re}\left[\epsilon_r(\omega')-1\right]\;\mathrm{d}\omega' = \frac{\omega_p^2}{\omega_L}\,. \tag{5.179}$$

For large ω_L, we must therefore have the property

$$(\mathrm{P.V.})\int_0^{\omega_L} \mathrm{Re}\;\epsilon_r(\omega')\;\mathrm{d}\omega' - \omega_L = \frac{\omega_p^2}{\omega_L} + \mathcal{O}(\omega_L^{-2})\,, \tag{5.180}$$

$$\frac{1}{\omega_L}\;(\mathrm{P.V.})\int_0^{\omega_L} \mathrm{Re}\;\epsilon_r(\omega')\;\mathrm{d}\omega' = 1 + \frac{\omega_p^2}{\omega_L^2} + \mathcal{O}(\omega_L^{-3})\,. \tag{5.181}$$

This is sometimes called a *superconvergence* relation.

In the evaluation of the integral, it is necessary to remember that the principal value integration can be done in three ways: (i) just do the integral analytically, then use the upper and lower boundaries of the integration interval, then throw away the imaginary part, (ii) if x is the integration variable and $x = a$ is the location of the singularity, then integrate up to $x = a - \epsilon$, and start again from $x = a + \epsilon$; throw away the divergent term in $1/\epsilon$, and (iii) drive a complex contour immediately above and immediately below the singularity on the real axis, and take the mean. The different sense of revolving around the singularity then means that its contribution cancels in the mean. For the current case, it means that we can take a contour in the upper half of the complex plane, and one in the lower half, and then close the contour in their respective halfs because of sufficient suppression of the integrand as $|\omega| \to \infty$.